Nuances of longitudinal dynamics in variable-tune linear field FFAG.

Topics :

1) Getting the energy range you want

2) Optimisation criterion

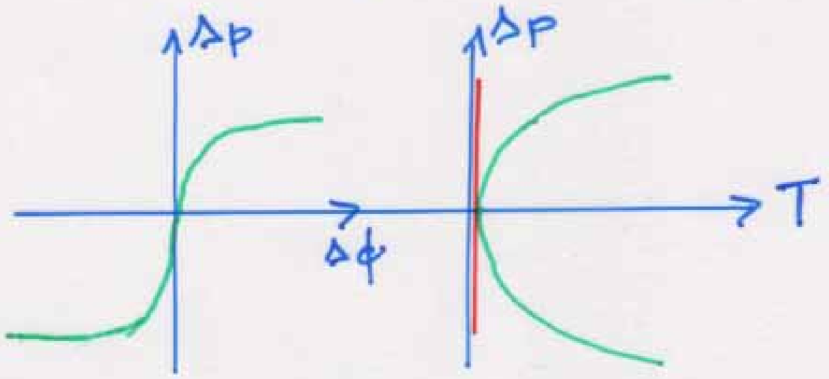
   i) dwell time — minimize τ

   ii) efficiency $\langle \cos\phi \rangle$ — maximize

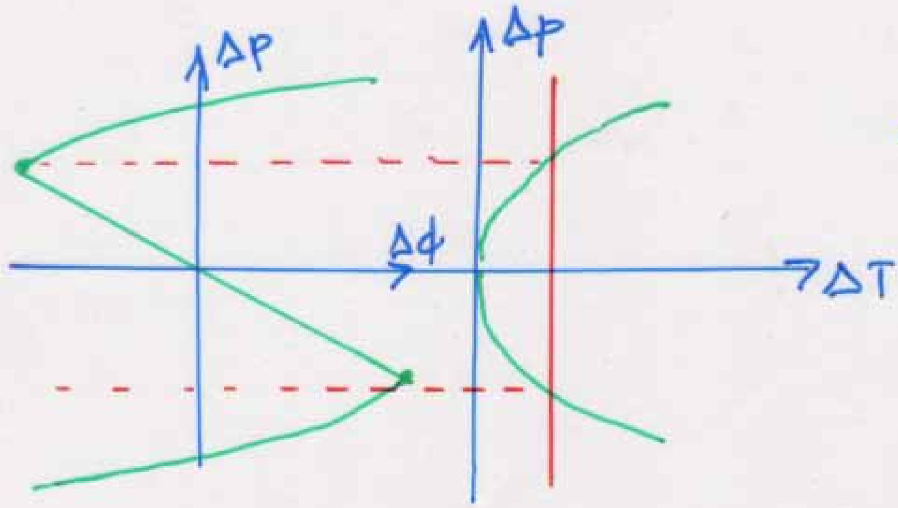
   iii) time dispersion $\frac{1}{2!}\frac{\partial^2 \tau}{\partial H^2}$ — minimize

   iv) match ellipse orientation angle + aspect ratio

3) How to choose precise value of RF w.r.t. ToF parabola?



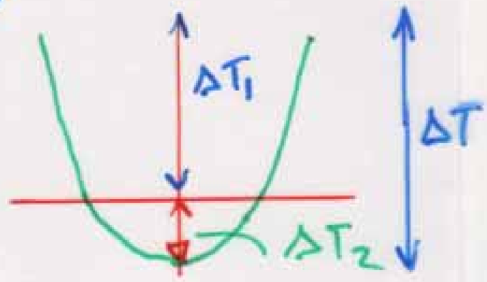
momentum range $\propto V^{1/3}$



at channel opening

momentum range between fixed points $\propto V$

Equations of motion



$$\frac{dy}{ds} = a.\cos x \qquad \frac{dx}{ds} = (2y)^2 - b$$

$$y = (E - \bar{E})/\Delta E \qquad x = \omega T$$

$$s = n\omega\Delta T \qquad a = \frac{\delta E}{\Delta E}\frac{1}{\omega \Delta T} = w \qquad b = \frac{\delta T_2}{\Delta T} = z$$

Nominal injection/extraction $\Rightarrow y = \mp \frac{1}{2}$ $\qquad \frac{a}{b^{3/2}} = \lambda$

a	b	$\pm\Delta y$
1/24	1/4	1/2
1/12	0	0.4
1/12	1/6	$\frac{1}{2}$
1/12	1/4	0.55
1/6	0	1/2
1/3	1	1

Q: Can we reverse engineer ΔE so that actual range equal nominal range?

A: Yes, but have only one free parameter α

$$\Delta T = (\alpha \Delta E)^2$$

$$\Delta E = \frac{\Delta E_n}{\Delta y}, \quad \Delta T_n = \frac{\Delta y^3}{a\omega}\frac{\delta E}{\Delta E_n}$$

b	Δy^3
1/6	1.0
1/5	1.124
1/4	1.323

$\therefore$ $b\downarrow$ implies permissable ToF variation smaller, lattice gets marginally more difficult

Dwell time τ, efficiency $\langle \cos x \rangle$, dispersion $\frac{1}{2!}\frac{\delta^2 \tau}{\delta H^2}$

requires us to compute integrals

values will depend on limits of integration

$$\tau = \int ds = \int_{\check{x}}^{\hat{x}} \frac{dx}{(dx/ds)} = \int_{\check{y}}^{\hat{y}} \frac{dy}{(dy/ds)}$$

expectation value $\langle f \rangle \tau = \int f \frac{dx}{(dx/ds)} = \int f \frac{dy}{(dy/ds)}$

$$\therefore a\tau \langle \cos x \rangle = (\hat{y} - \check{y}).$$

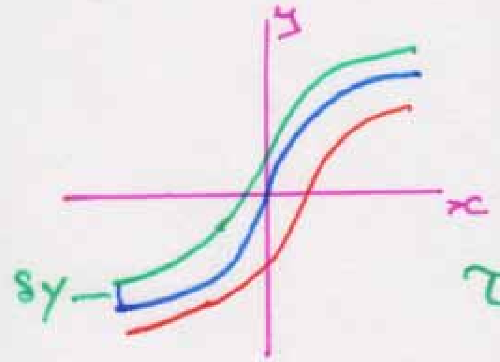
General case $\Rightarrow$ numerical integration BUT singularities!

Special cases $\Rightarrow$ useful or illustrative.

RF period synchronous with bottom of parabola

i.e $\delta T_2 = 0$ Integration range $x = \pm \pi/2$

$$\tau = \frac{2.2066}{a^{2/3}} \qquad \langle \cos x \rangle = 0.82350$$



Reference trajectory passes thru $(x, y) = (0, 0)$

h = hamiltonian value = zero.

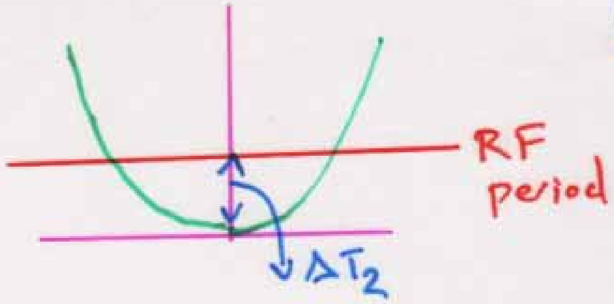
$$\tau(h) = \tau(0)\left[1 + \frac{2}{9}\left(\frac{h}{a}\right)^2 + \cdots\right]$$

$$\tau(\delta y) = \tau(0)\left[1 + \frac{2}{9}\left(\frac{\delta y}{a}\right)^2 (6a)^{4/3} + \cdots\right]$$

$\Rightarrow$ smallest possible value of time dispersion.

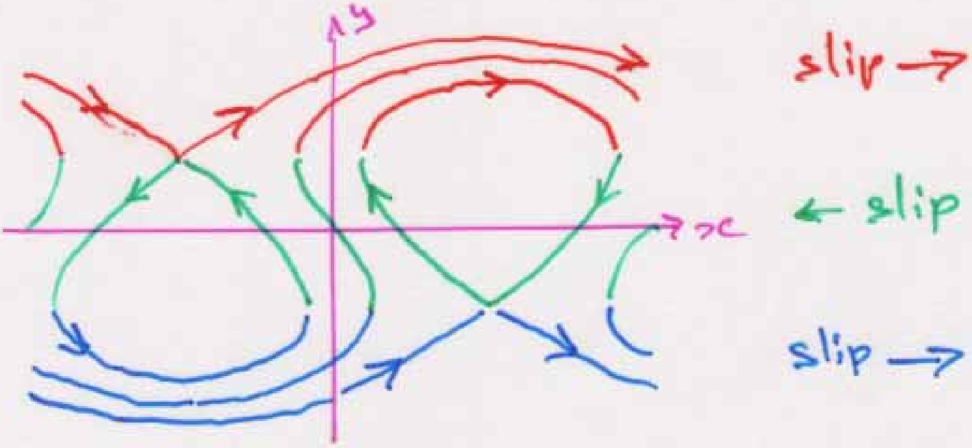
Can also compute τ, $\langle \cos x \rangle$, $\frac{\delta \tau}{\delta h^2}$ for integration range $y = \pm 1/2$

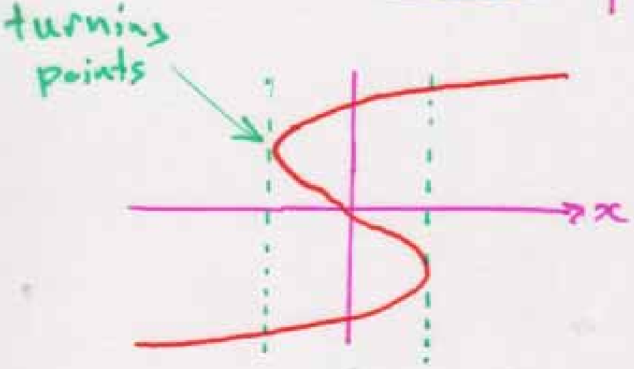
Two slip reversals, $\Delta T_2 \neq 0$

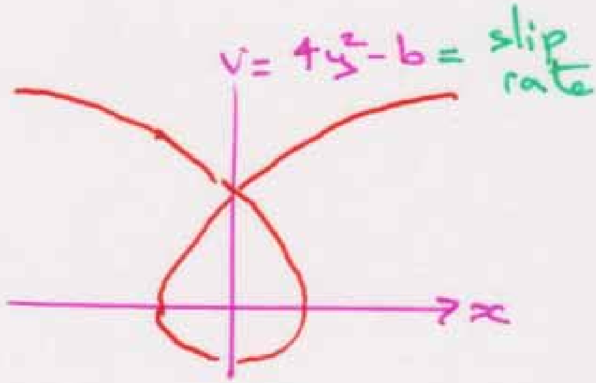


hamiltonian:

$(4/3)y^3 - yb - a\sin x$







Provided that NOT $a \gg b^{3/2}$

integrals are dominated by motion between turning points.

– because slip rate is smallest in that range.

Serpentine channel opening condition is

$$a > a_c = \frac{b^{3/2}}{3}$$

On central trajectory (passes thru $x = y = 0$)

$$\tau = \frac{4\sqrt{b}}{a\pi} K\left[\frac{a_c^2}{a^2}\right] \qquad \langle \cos x \rangle = \frac{\pi/2}{K[\ldots]} \to \text{unity}$$

↑ complete elliptic integral 1st kind.

dispersion in arrival time for particles off central trajectory

$$\tau(h) = \tau(0) + \frac{2\sqrt{b}\,c^2}{a(a^2-a_c^2)^2}\left[2a^2\,\underset{\uparrow}{\mathcal{E}} + (a_c^2-a^2)\,\mathcal{K}\right] + \cdots$$

complete elliptic integral 2nd kind.

Significant feature is the resonant denominator terms $(a^2-a_c^2) \Rightarrow$ strong dispersion $(a^2 - b^3/9)$

For given a, the larger is b the closer motion is to fixed point.

Slightly subjective conclusion, best operating point is $(a,b) = (\frac{1}{12}, \frac{1}{6})$

Variation of τ with hamiltonian is key to minimizing emittance distortion due to nonlinearity.

Ideally all points on boundary of phase-space ellipse have same |hamiltonian|

$\Rightarrow$ conditions for aspect ratio (ΔE vs $\Delta\phi$) and orientation angle if inject away from $x = -\pi/2$.

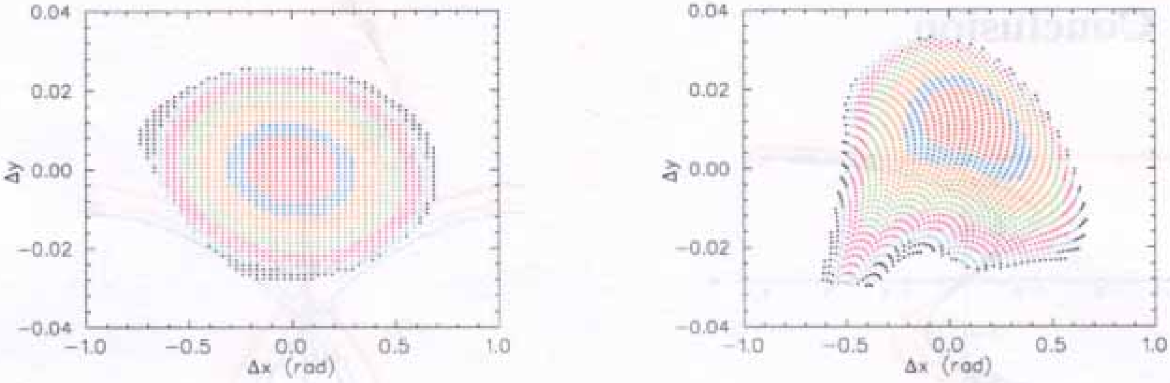


Figure 24: Input (left) and output (right) phase space. $(a, b) = (1/12, 1/6)$, tracking range $x = \pm\pi/2$, 91.1% survival.

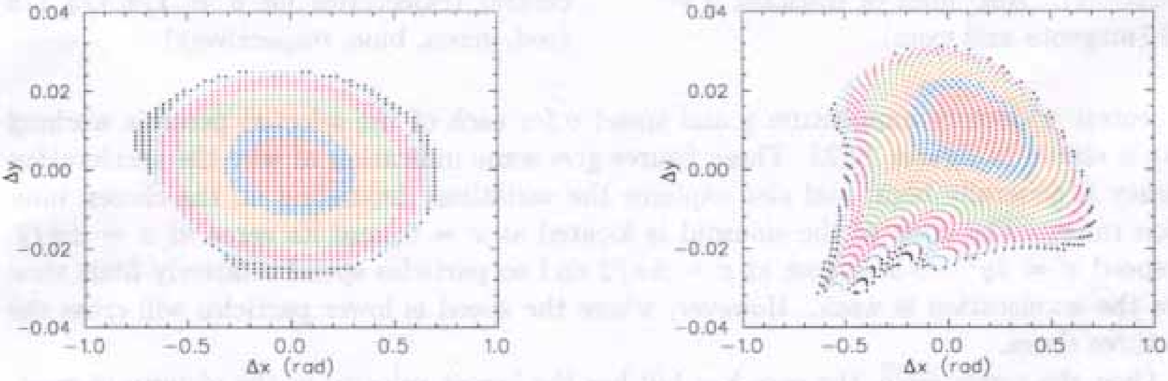


Figure 25: Input (left) and output (right) phase space. $(a, b) = (1/12, 1/5)$, tracking range $x = \pm\pi/2$, 90.2% survival.

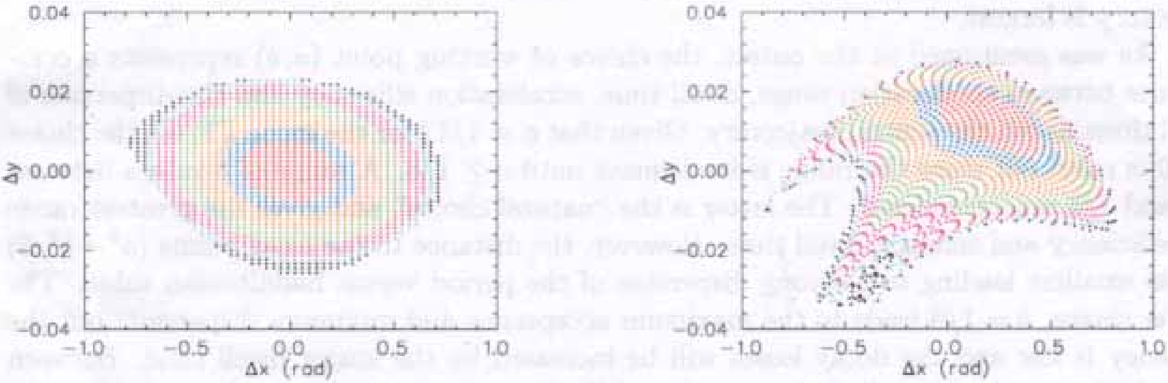


Figure 26: Input (left) and output (right) phase space. $(a, b) = (1/12, 1/4)$, tracking range $x = \pm\pi/2$, 84.1% survival.

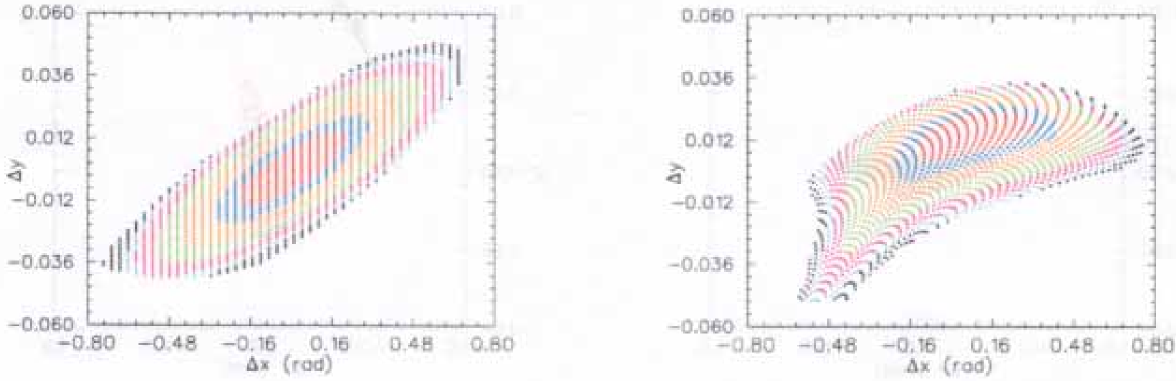


Figure 27: Input (left) and output (right) phase space. $(a, b) = (1/12, 1/5)$, tracking range $y = \pm 1/2$, 87.4% survival.

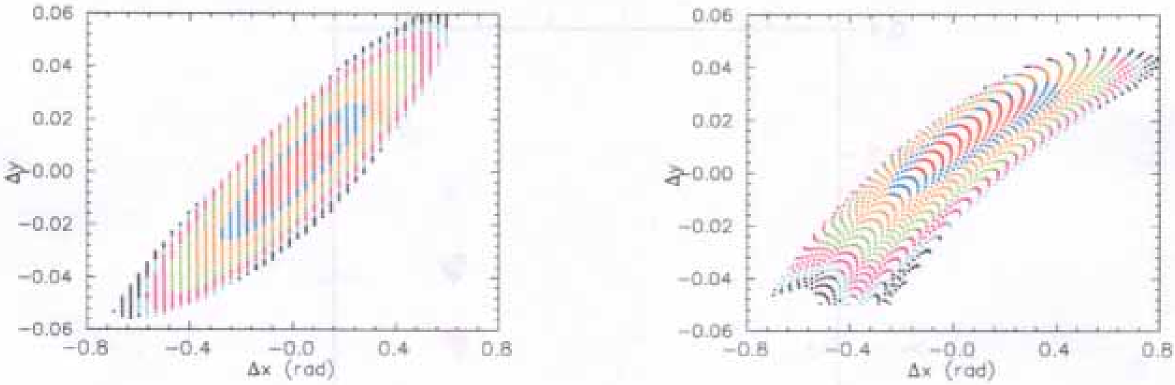


Figure 28: Input (left) and output (right) phase space. $(a, b) = (1/12, 1/4)$, tracking range $y = \pm 1/2$, 76.0% survival.

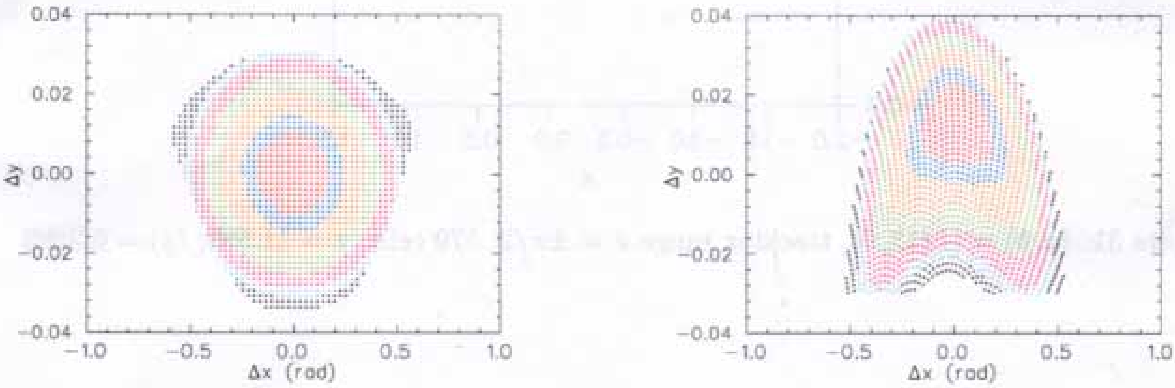


Figure 29: Input (left) and output (right) phase space. $(a, b) = (1/6, 0)$, tracking range $x = \pm\pi/2$, 85.7% survival.