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Basic Beam Optics

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Outline

- Relativistic dynamics, relativistic beta & gamma
- Gradient (weak) and alternating (strong) focusing
  - dipole, quadrupole, sextupole magnets
- Mathieu-Hill equation, Courant-Snyder parameters
- Particle beams, emittance
- Betatron tunes, chromaticity, resonances
- Longitudinal basics, momentum compaction, slip factor
- Synchronous acceleration, longitudinal Hamiltonian, phase stability
- Intensity effects and space charge driven resonance

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Notation and Basic Formulae

Velocity of light	$c = 2.99792458 \times 10^8$ m/sec
Relative velocity	$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \boldsymbol{\beta} c$
Relativistic gamma	$\gamma = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}}$
Rest mass	m_0
Relativistic mass	$m = m_0 \gamma$
Momentum	$\mathbf{p} = m\mathbf{v} = m_0 \gamma \mathbf{v} = m_0 \gamma \boldsymbol{\beta} c$
Energy	$\mathcal{E} = mc^2 = m_0 \gamma c^2$
Kinetic energy	$T = \mathcal{E} - m_0 c^2 = m_0(\gamma - 1)c^2$

Note: $$\frac{\mathcal{E}^2}{c^2} = \mathbf{p}^2 + m_0^2 c^2$$

Maxwell's Equations

$\vec{E}$ Electric field

$\vec{B}$ Magnetic flux density

ρ Charge density

$\vec{j}$ Current density

μ_0 Permeability of free space, $4\pi \times 10^{-7}$

ϵ_0 Permittivity of free space, 8.854×10^{-12}

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

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Review of Simple Particle Dynamics

Equation of motion in electromagnetic fields:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\Big(\mathbf{E} + \mathbf{v} \times \mathbf{B}\Big)$$

Lorentz Force

produces acceleration (→ $\mathbf{E}$) produces bending (→ $\mathbf{B}$)

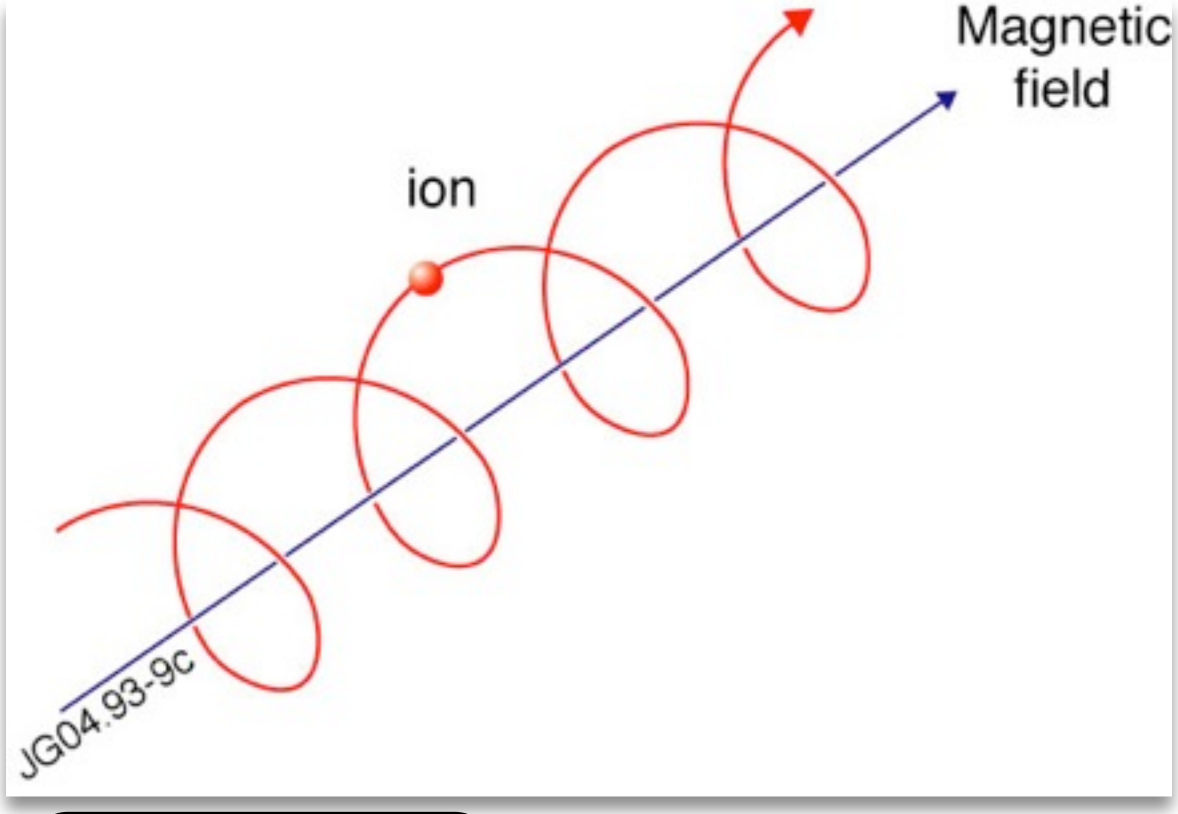
Total energy $\mathcal{E} = mc^2$ and $\mathcal{E}^2 = \mathbf{p}^2 c^2 + m_0^2 c^4$

$$\Longrightarrow \qquad \mathcal{E}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = c^2\mathbf{p}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = qc^2\mathbf{p}\cdot\mathbf{E}$$

$$\Longrightarrow \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\mathbf{v}\cdot\mathbf{E}$$

Energy change only from Electric fields

Motion in Constant Magnetic Field



$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\vec{v}) = q\vec{v}\wedge\vec{B}$$

$$\implies \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m_0\gamma}\vec{v}\wedge\vec{B}$$

$$\implies \frac{v_\perp^2}{\rho} = \frac{q}{m_0\gamma}v_\perp B$$

$\implies$ circular motion with radius $\rho = \dfrac{m_0\gamma v_\perp}{qB}$

at an angular frequency $\omega = \dfrac{v_\perp}{\rho} = \dfrac{qB}{m_0\gamma} = \dfrac{qB}{m}$

Constant magnetic field gives uniform spiral about B with constant energy.

$$B\rho = \frac{m_0\gamma v}{q} = \frac{p}{q}$$

Magnetic Rigidity

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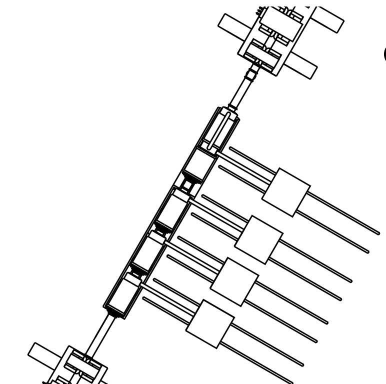
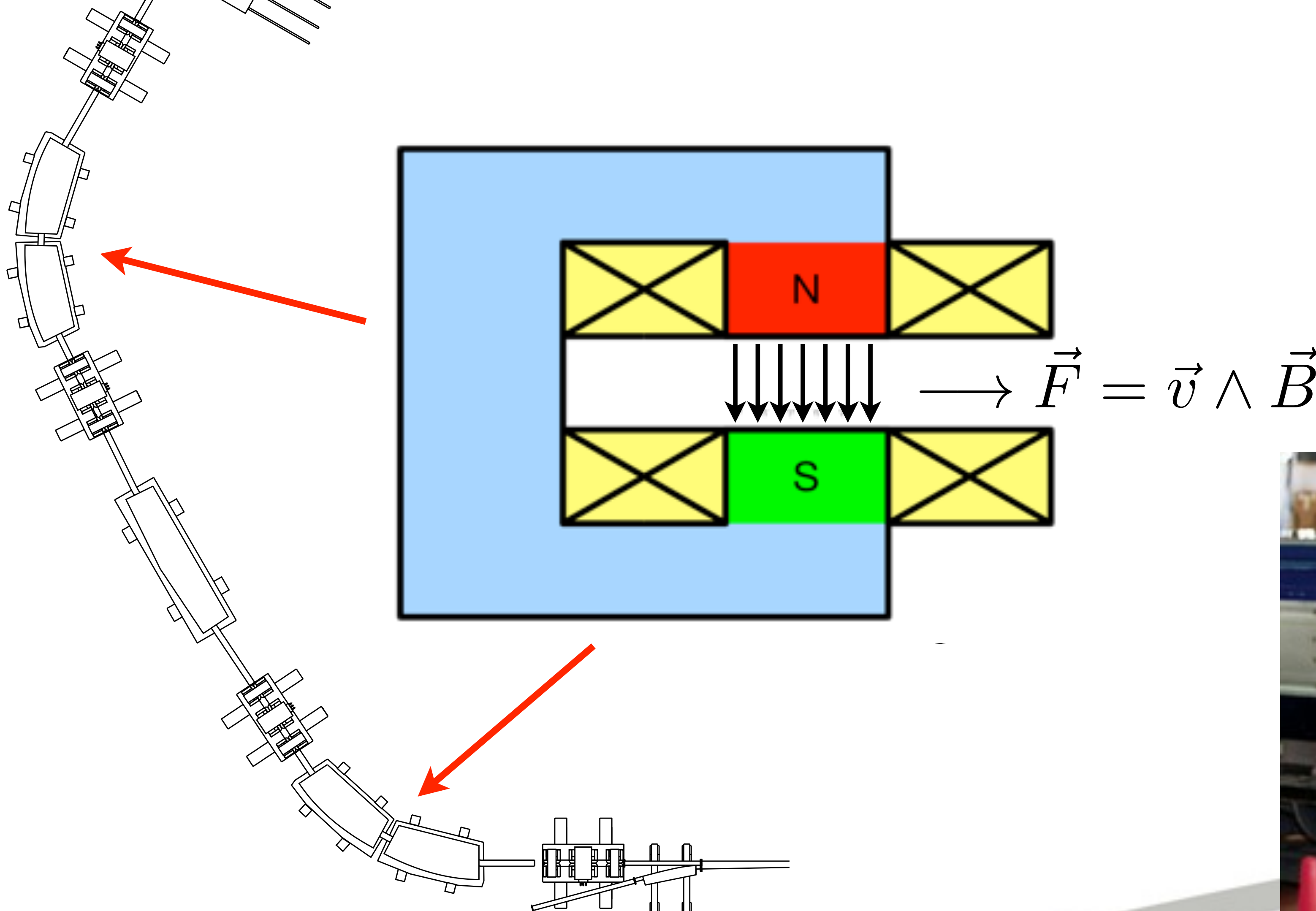
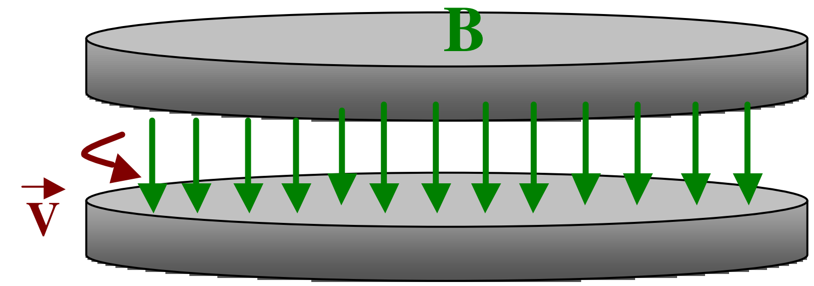
Vertical Magnetic Field

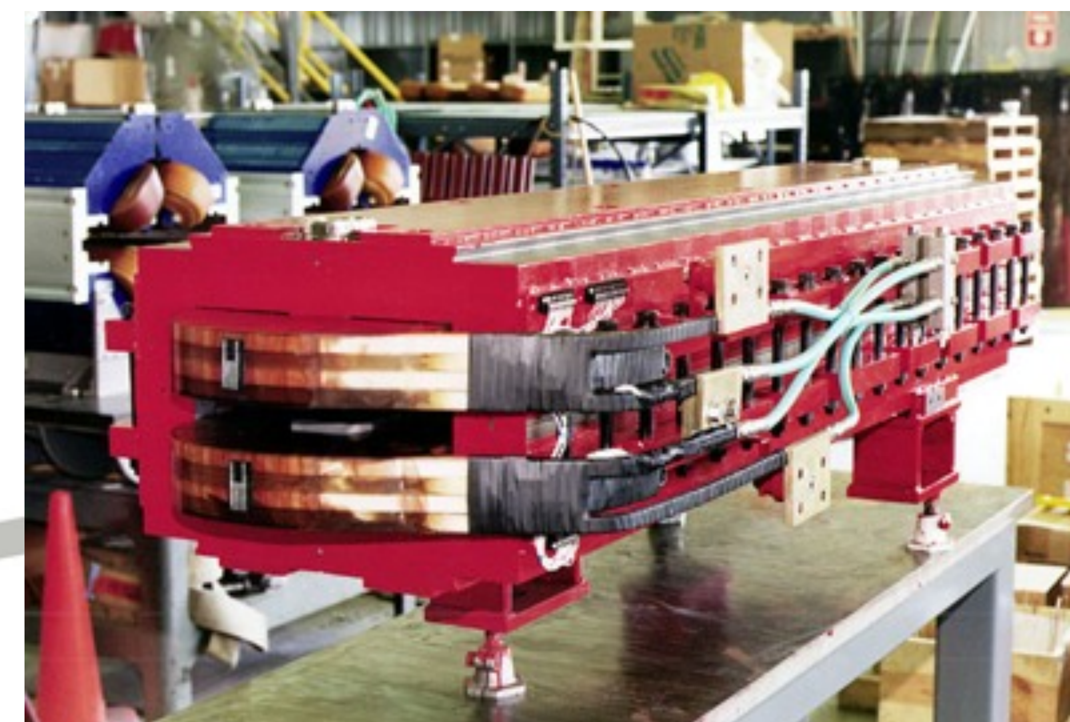
- Dipole magnet:
  - a vertical magnetic field can be used to bend particles and maintain their path in a circular accelerator





$$\longrightarrow \vec{F} = \vec{v} \wedge \vec{B}$$

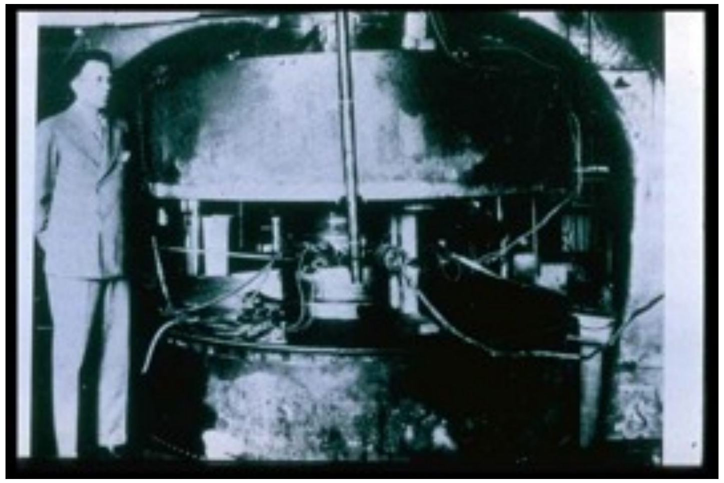




Example: Cyclotron

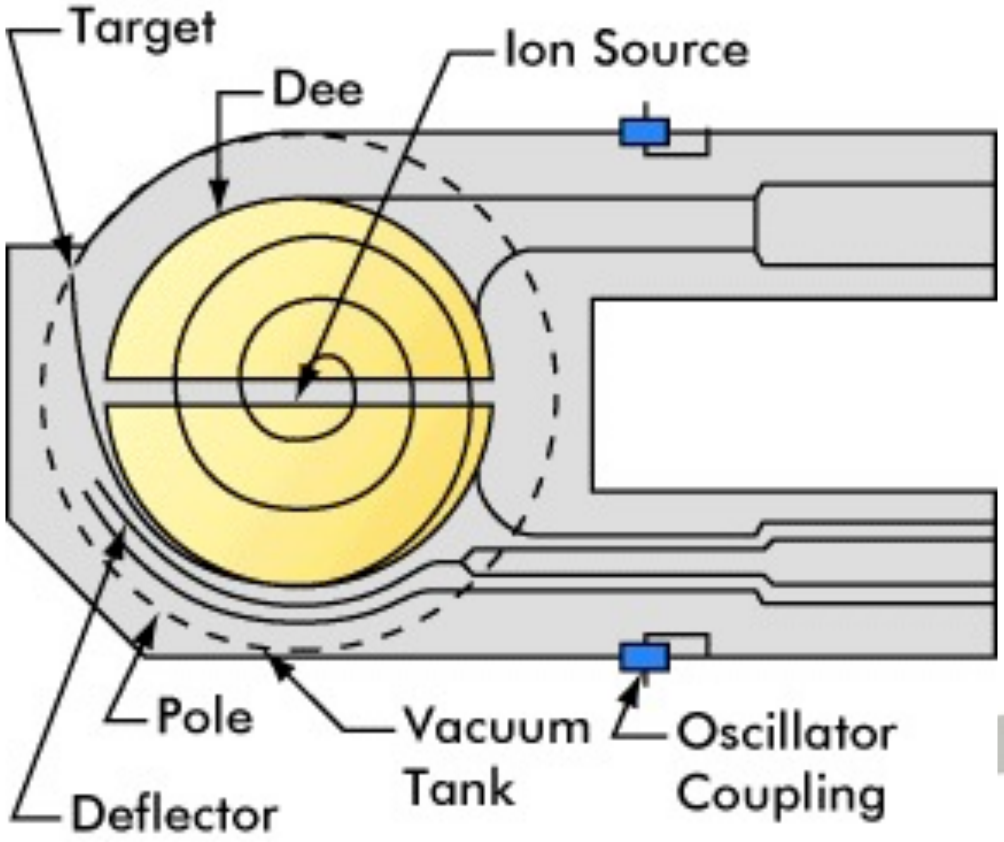
$$\rho = \frac{p}{qB}, \qquad \omega = \frac{v}{\rho} = \frac{qB}{m} = \frac{qBc^2}{\mathcal{E}}$$

$$f = n\omega, \quad n \in \mathbb{N}$$



Ernest Lawrence and cyclotron, 1932

- Use magnetic fields to force particles to circulate and pass through accelerating fields at regular intervals
  - Constant B field
  - Constant accelerating frequency f
  - Spiral trajectories
  - For synchronism $f = n\omega$, which is possible only at low energies, $\gamma \sim 1$.
  - Use for heavy particles (protons, deuterons, α-particles).



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Methods of Acceleration: Circular

$$\rho = \frac{p}{qB}$$

$$f = n\omega$$

$$\omega = \frac{qBc^2}{\mathcal{E}} = \frac{v}{\rho}$$

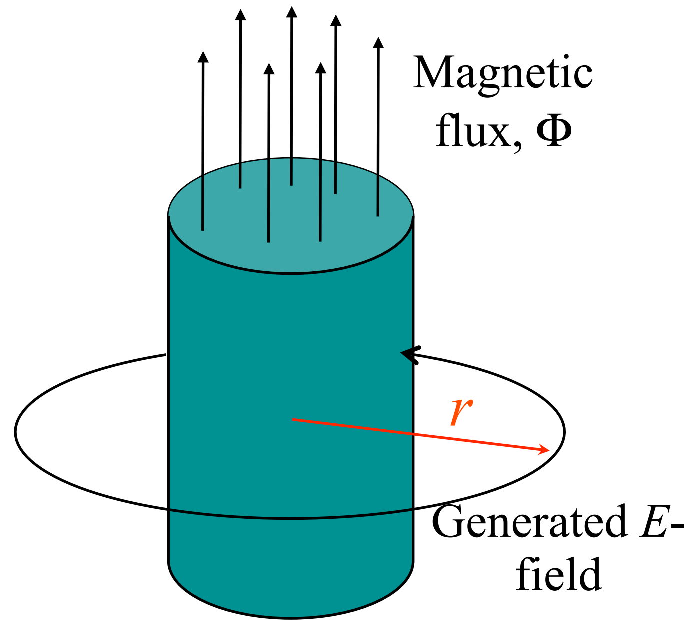
- Higher energies => relativistic effects => ω no longer constant.
- Particles get out of phase with accelerating fields; eventually no overall acceleration.
- **Isochronous cyclotron**
  - Vary *B* to compensate and keep f constant.
  - For stable orbits need both radial (because ρ varies) and azimuthal *B*-field variation
  - Leads to construction difficulties.
- **Synchro-cyclotron**
  - Modulate frequency *f* of accelerating structure instead.
  - In this case, oscillations are stable (McMillan & Veksler, 1945)
- **Synchrotron**
  - Principle of frequency modulation but in addition vary B-field in time to match increase in energy and keep revolution radius constant.

The Betatron

Particles accelerated by the rotational electric field generated by a time-varying magnetic field

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \vec{B} \cdot \mathrm{d}\vec{S}$$

$$\Longrightarrow \quad 2\pi r E_\theta = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

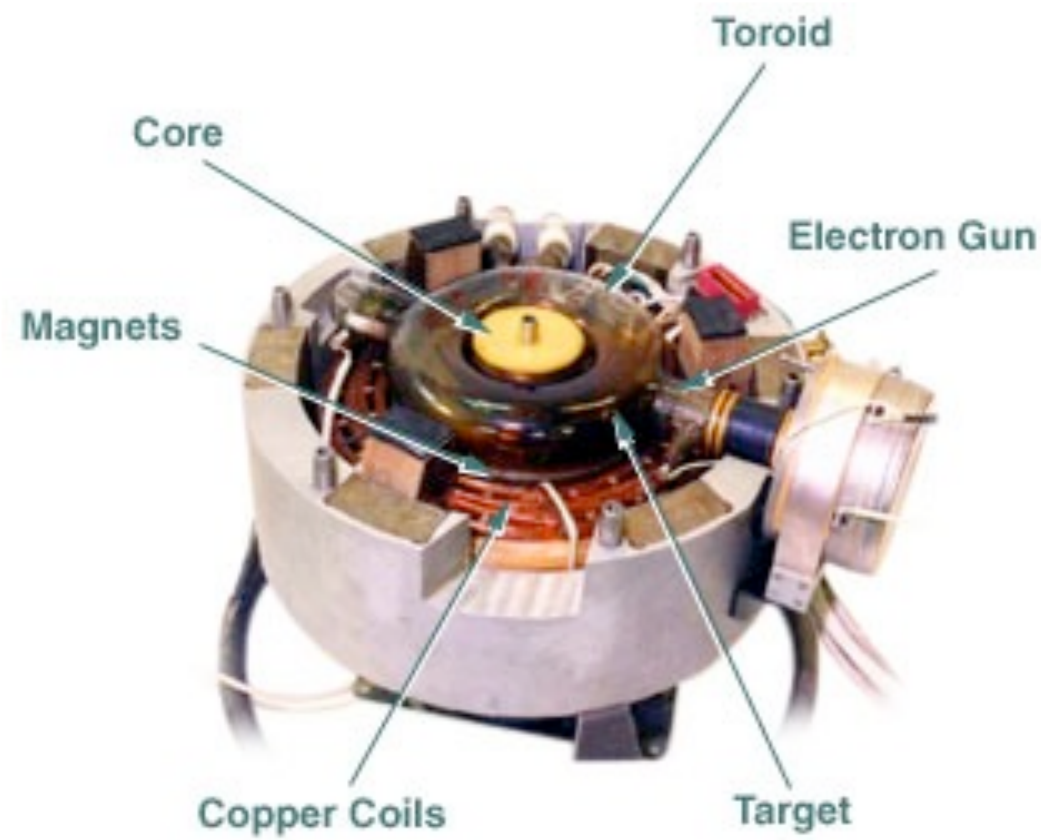


For circular motion at a constant radius:

$$-\frac{mv^2}{r} = evB \quad \Longrightarrow \quad B = -\frac{p}{er}$$

$$\Longrightarrow \frac{\partial}{\partial t} B(r,t) = -\frac{1}{er}\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{E}{r} = \frac{1}{2\pi r^2}\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

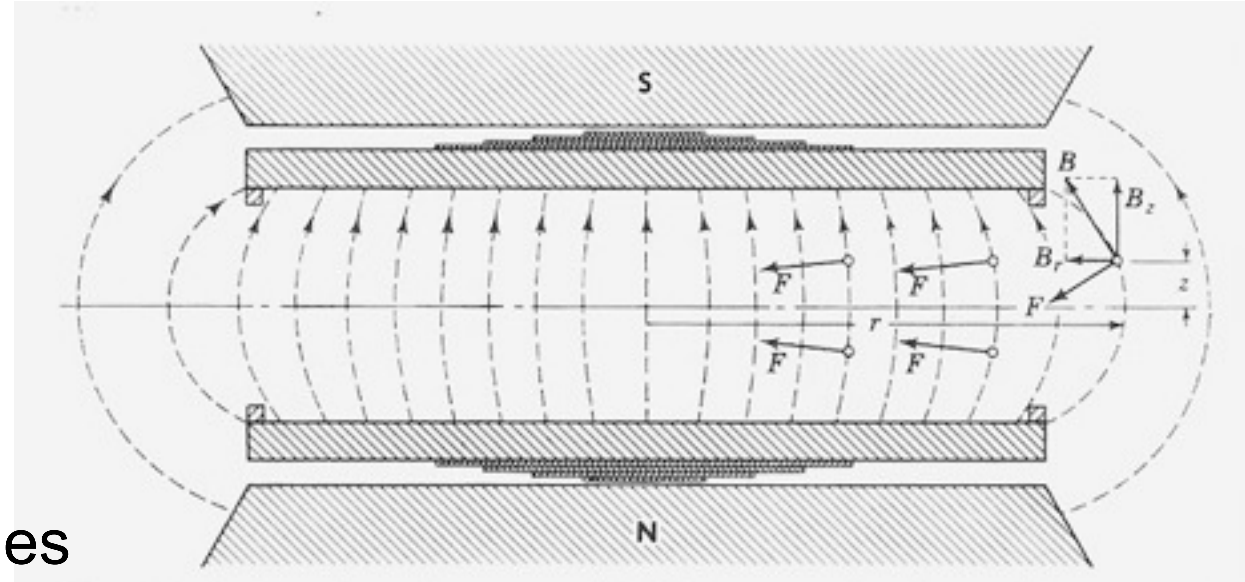
$$\Longrightarrow B(r,t) = \frac{1}{2}\frac{1}{\pi r^2}\iint B \,\mathrm{d}S$$



B-field on orbit needs to be one half the average *B* over the circle. This imposes a limit on the energy that can be achieved. Nevertheless the constant radius principle is attractive for high energy circular accelerators.

Transverse Control: Weak Focusing

- Particles injected horizontally into a uniform, vertical, magnetic field follow a circular orbit.
- Misalignment errors and difficulties in perfect injection cause particles to drift vertically and radially and to hit walls.
- Require some kind of stability mechanism.
- Design source of magnetic field so that field lines bend outwards
- Vertical focusing from non-linearities in the field (fringing fields). Vertical stability requires negative field gradient. But this is at the expense of radial focusing, so effectiveness of the overall focusing is limited.



$$qvB = \frac{mv^2}{\rho} > \frac{mv^2}{r} \quad \text{if} \quad r > \rho$$

i.e. horizontal restoring force is towards the design orbit.

$$\text{Stability condition:} \qquad 0 < n = -\frac{\rho}{B}\frac{\mathrm{d}B}{\mathrm{d}\rho} < 1$$

Weak focusing: if used, scale of magnetic components of a synchrotron would be large and costly

Strong Focusing

- Maxwell's time-independent equations, no sources:

$$\nabla \wedge \vec{B} = 0 \quad \Longrightarrow \quad \exists \phi \text{ such that } \vec{B} = \nabla \phi \quad \text{Scalar magnetic potential}$$

$$\nabla \cdot \vec{B} = 0 \quad \Longrightarrow \quad \nabla^2 \phi = 0, \quad \text{Laplace's equation}$$

- Simplest solutions, with $z = x + iy \quad (i = \sqrt{-1})$ are

$$\phi = K z^n, \;\; K \text{ constant}$$

$$\begin{array}{lll} n = 1 & \phi \propto x + iy & \text{dipole} \\ n = 2 & \phi \propto (x^2 - y^2) + 2ixy & \text{quadrupole} \\ n = 3 & \phi \propto x(x^2 - 3y^2) + iy(3x^2 - y^2) & \text{sextupole} \end{array}$$

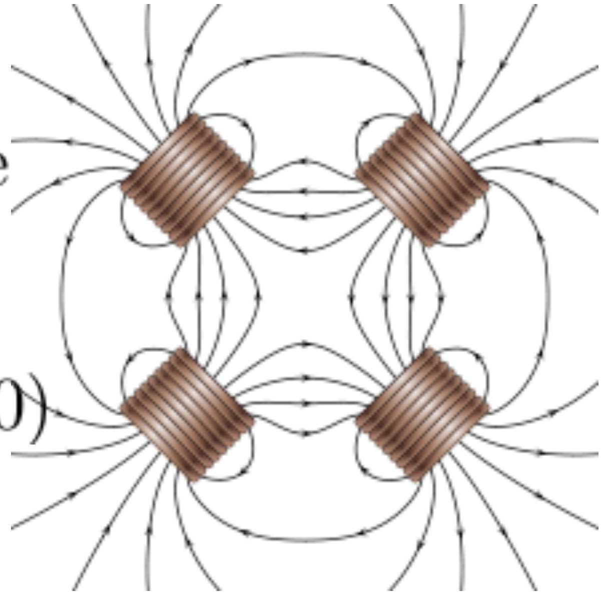
- Then $$\vec{B} = nK(1, i, 0) z^{n-1} \qquad \text{(real part understood)}$$

Focusing Elements

Quadrupole

$\phi \propto xy,$

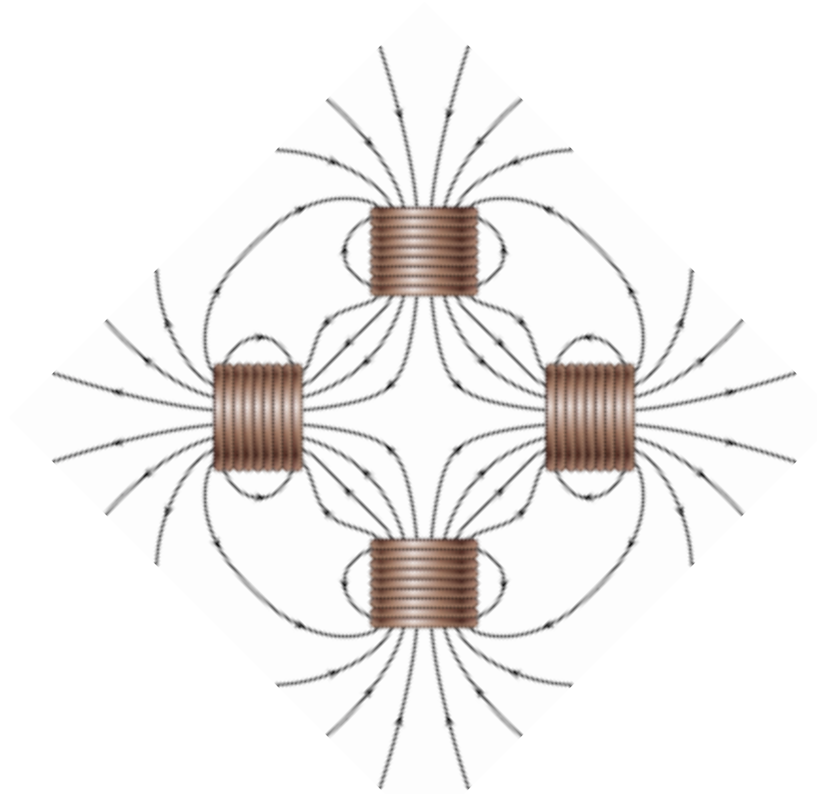
$\vec{B} = K(y, x, 0)$



Sextupole

$\phi \propto y(3x^2 - y^2),$

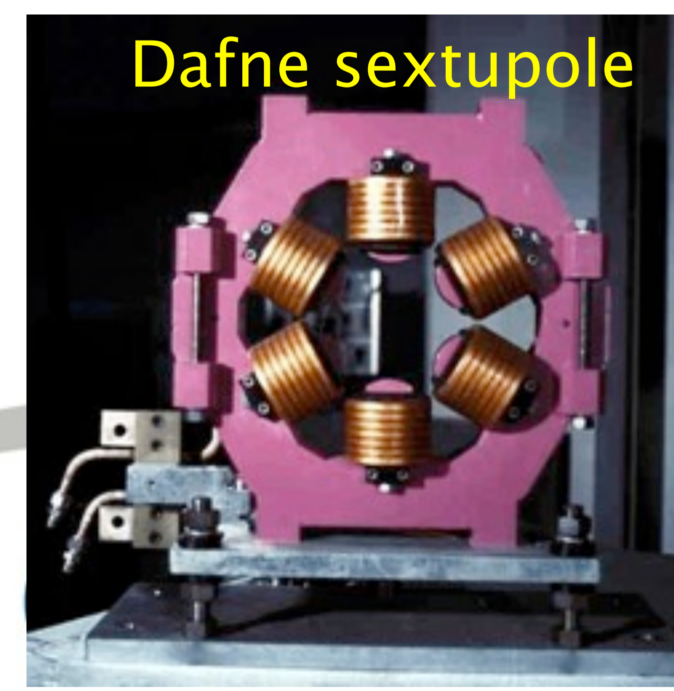
$\vec{B} = K(2xy, x^2 - y^2, 0)$



Skew quadrupole

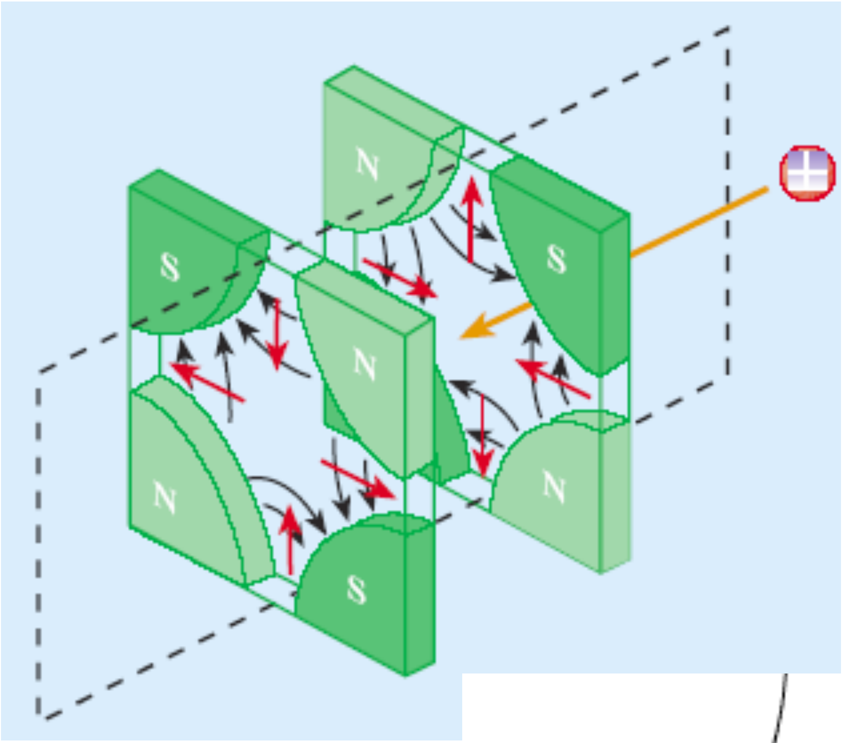
$\phi \propto x^2 - y^2,$

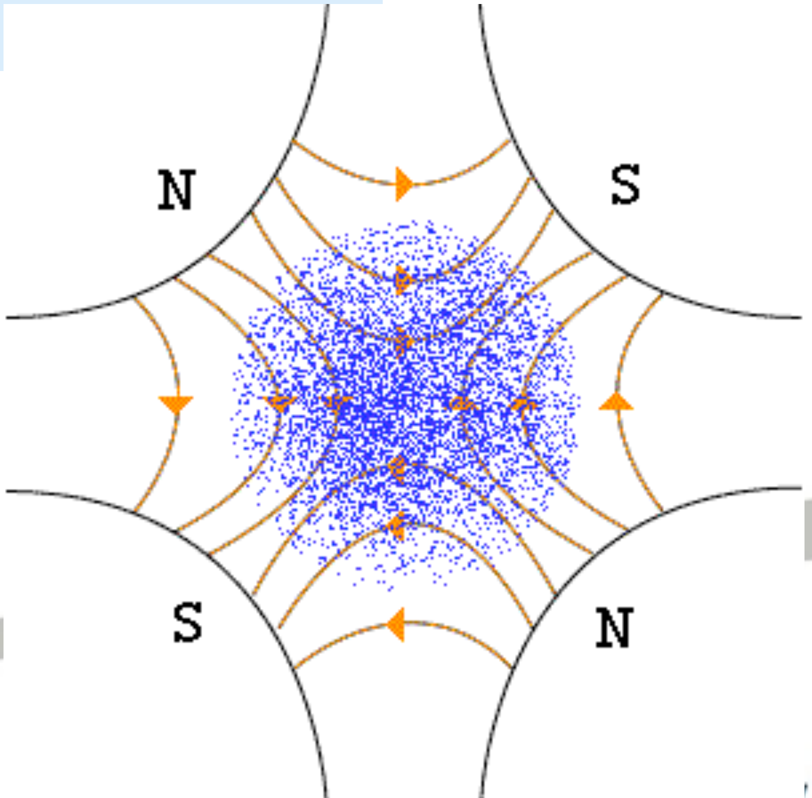
$\vec{B} = K(x, -y, 0)$



Strong Focusing: Alternating Gradient Principle

- A sequence of focusing-defocusing quadrupole fields provides a stronger net focusing force.
- Quadrupoles focus horizontally, defocus vertically or vice versa. Forces are linearly proportional to displacement from axis.
- A succession of opposed elements enable particles to follow stable trajectories, making small (betatron) oscillations about the design orbit.
- Technological limits on magnets are high.

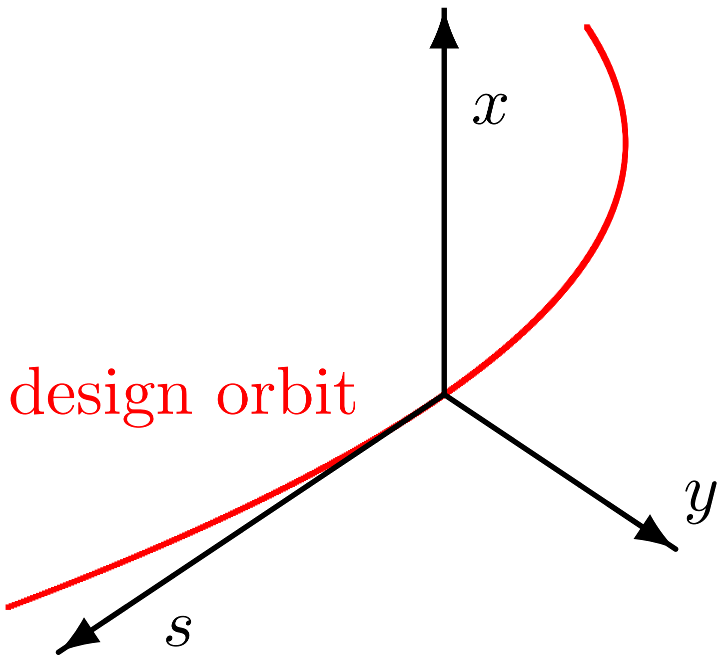




Strong focusing was first conceived by **Nicholas Christofilos** in 1949 but not published (Christofilos opted instead to patent his idea), and was later independently invented in 1952 at Brookhaven National Laboratory. The advantages of strong focusing were then quickly realised, and deployed on the Alternating Gradient Synchrotron (AGS). Courant, Livingston, Snyder and Blewett later acknowledged the priority of Christofilos' idea.

Equations of Motion

No bends



$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma\vec{v}) = q\vec{v} \wedge \vec{B} = q \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{s} \end{bmatrix} \wedge Kn \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} z^{n-1}$$

$$\Longrightarrow \qquad m_0 \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \gamma\dot{x} \\ \gamma\dot{y} \\ \gamma\dot{s} \end{bmatrix} = Knq \begin{bmatrix} -i\dot{s} \\ \dot{s} \\ i\dot{x} - \dot{y} \end{bmatrix} z^{n-1}$$

Write $x' = \dfrac{\dot{x}}{\dot{s}}, y' = \dfrac{\dot{y}}{\dot{s}}$, and note in an accelerator $\dot{x}, \dot{y} \ll \dot{s}$

so that $\mathbf{v} = (x', y', 1)\dot{s}, \quad x', y' \ll 1$ and $\dfrac{\mathrm{d}}{\mathrm{d}t} = \dot{s}\dfrac{\mathrm{d}}{\mathrm{d}s}$

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Paraxial Equations

From above, $\frac{\mathrm{d}}{\mathrm{d}t}(\gamma \dot{s}) \sim 0$

$$\gamma = \left(1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{s}^2}{c^2}\right)^{-\frac{1}{2}} \approx \left(1 - \beta^2(x'^2 + y'^2 + 1)\right)^{-\frac{1}{2}}$$

$$= (1 - \beta^2)^{-\frac{1}{2}} + \text{second order terms}$$

Ignoring second and higher velocity terms, equations of motion are

$$m_0 \gamma \beta^2 c^2 \begin{bmatrix} x'' \\ y'' \end{bmatrix} = K n q \beta c \begin{bmatrix} -i \\ 1 \end{bmatrix} z^{n-1}$$

$n = 2$ (K imag)	quadrupole	$\begin{bmatrix} x'' + kx \\ y'' - ky \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$n = 2$ (K real)	skew quadrupole	$\begin{bmatrix} x'' + ky \\ y'' + kx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$n = 3$ (K imag)	sextupole	$\begin{bmatrix} x'' + k(x^2 - y^2) \\ y'' - 2kxy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$n = 3$ (K real)	skew sextupole	$\begin{bmatrix} x'' + 2kxy \\ y'' + k(x^2 - y^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$k = q \times \frac{\text{Field strength}}{m_0 \gamma \beta c}$$

$$= \text{Field strength}/B\rho$$

Equations of Motion in Dipoles

Need to take into account the change in direction of the axes $\Longrightarrow \quad \dot{s} \to \dot{s}(1+\kappa x)$ where $\kappa = \dfrac{1}{\rho}$ is the curvature of the design orbit.

See appendix

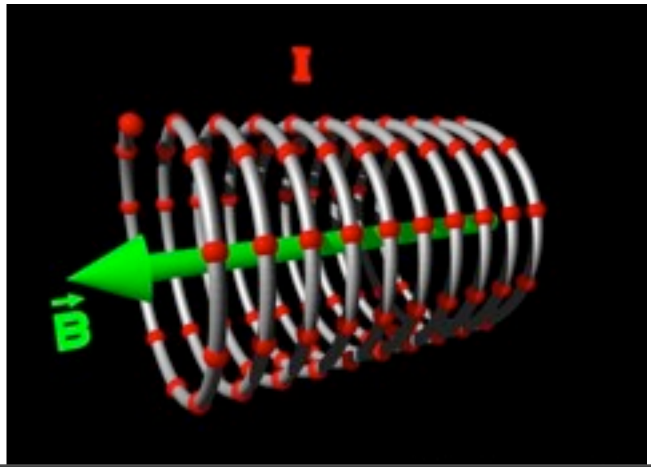
Equations become

$$\begin{bmatrix} x'' + \kappa^2 x \\ y'' \end{bmatrix} = \begin{bmatrix} x'' + \dfrac{1}{\rho^2} x \\ y'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

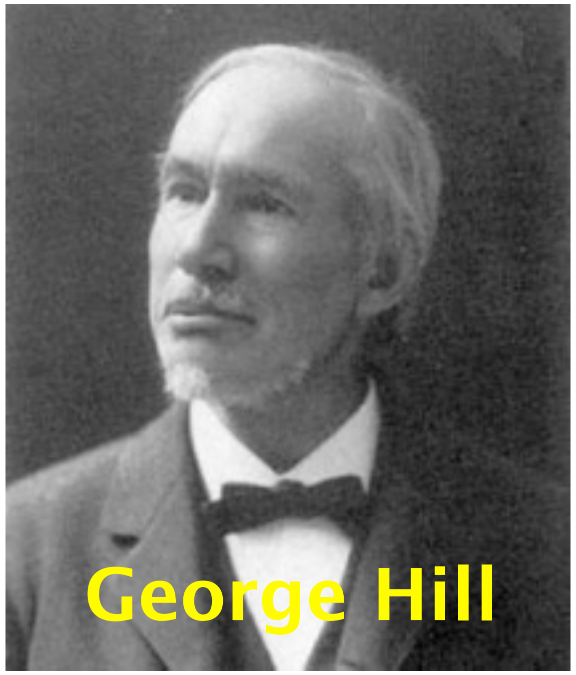
So dipole acts like a focusing quadrupole in the plane of the bend.

Also note solenoids:

$$\begin{bmatrix} x'' + 2ky' + k'y \\ y'' - 2kx' - k'x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Mathieu-Hill Equations



Paraxial equation of motion in periodic systems:

$$\begin{aligned} x''(s) + k_x(s)x &= 0 \\ y''(s) + k_y(s)y &= 0 \end{aligned}$$

where s is distance along beam axis

$k_x(s)$, $k_y(s)$ periodic focusing functions, $k(s+L) = k(s)$

Floquet's Theorem confirms two independent solutions:

$$u = w(s)\mathrm{e}^{i\psi(s)}, \quad v = w(s)\mathrm{e}^{-i\psi(s)}$$

The Wronskian is $W(u,v) = uv' - vu' = -2iw^2\psi' = C$, a constant

$$\text{Choose } C = -2i \quad \Longrightarrow \quad \frac{\mathrm{d}\psi}{\mathrm{d}s} = \psi' = \frac{1}{w^2}.$$

Then, substitute u or v into Mathieu-Hill equation:

$$u' = (w' + iw\psi')\mathrm{e}^{i\psi} = (w' + i/w)\mathrm{e}^{i\psi},$$

$$u'' = (w'' - iw'/w^2 + iw'\psi' - \psi'/w)\mathrm{e}^{i\psi} = (w'' - 1/w^3)\mathrm{e}^{i\psi}$$

$$u'' + ku = 0 \quad \Longrightarrow \quad \boxed{w'' + kw - \frac{1}{w^3} = 0} \qquad (1)$$

Any solution of Matthieu-Hill is a linear combination of u, v,

so set $\boxed{x = A\,w(s)\,\cos(\psi(s) + \phi).}$

$$\frac{x}{w} = A\cos(\psi + \phi), \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x}{w}\right) = -\frac{A}{w^2}\sin(\psi + \phi)$$

$$\Longrightarrow \qquad \frac{x^2}{w^2} + \Big(wx' - w'x\Big)^2 = A^2.$$

Or: $\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2}$

where $\beta = w^2, \quad \alpha = -ww' = -\frac{1}{2}\beta', \quad \gamma = \frac{1}{w^2} + w'^2 = \frac{1 + \alpha^2}{\beta}$

Phase-Space Ellipse; Emittance

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = A^2$$

Area of ellipse is $\pi A^2(\beta\gamma - \alpha^2) = \pi A^2$

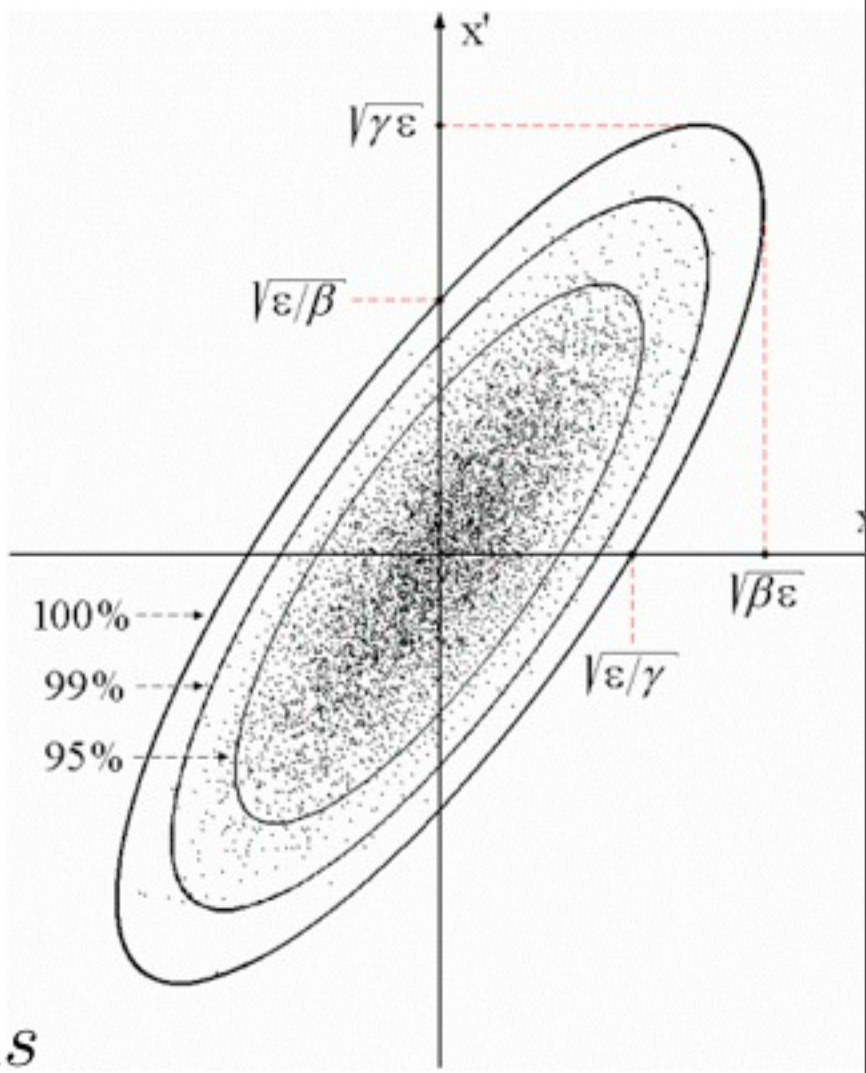
Area of largest ellipse for **all particles in beam** is denoted $\pi\epsilon$

$\gamma x^2 + 2\alpha x x' + \beta x'^2 \leq \epsilon$ is beam ellipse in x-x' phase space and ϵ is called the **beam emittance**

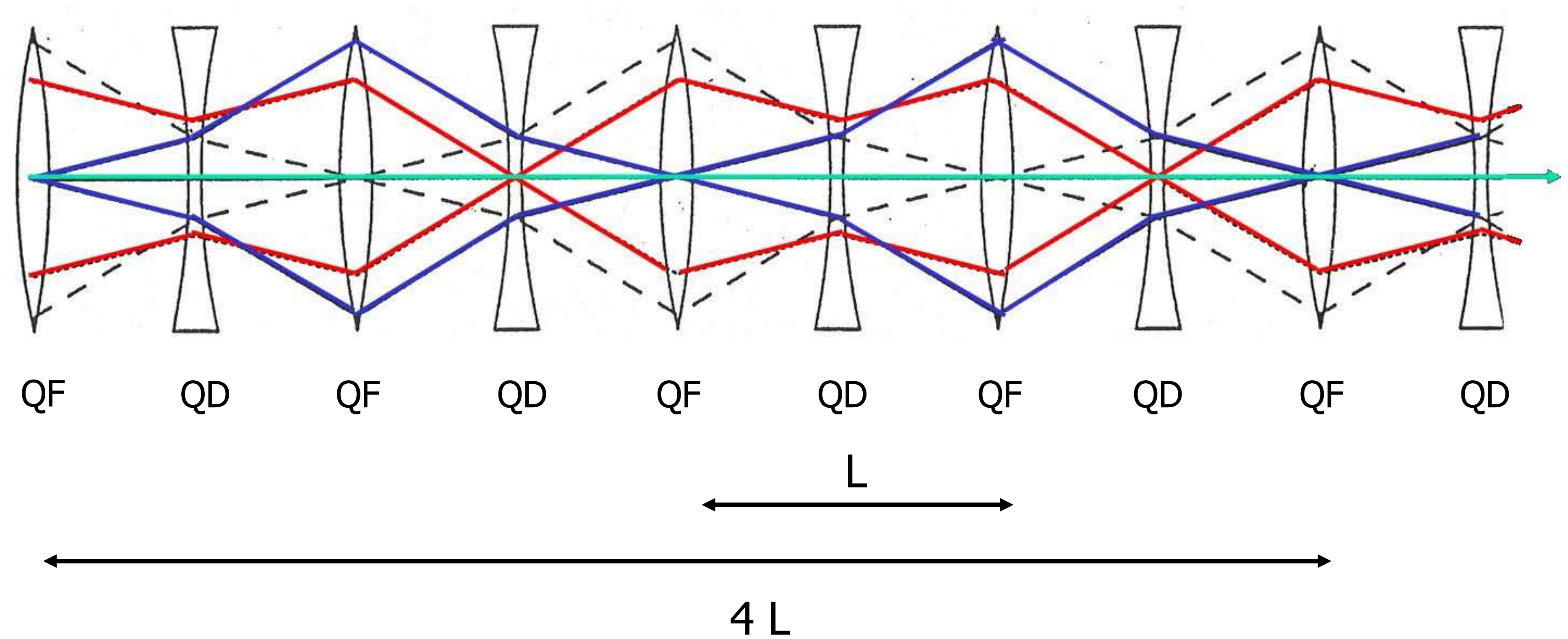
α, β, γ are Courant-Snyder parameters

Beam size (half-width) is $a(s) = \sqrt{\epsilon\beta(s)}$

Phase advance $\psi(s) = \int \psi' \, \mathrm{d}s = \int \frac{\mathrm{d}s}{w^2} = \int \frac{\mathrm{d}s}{\beta}$

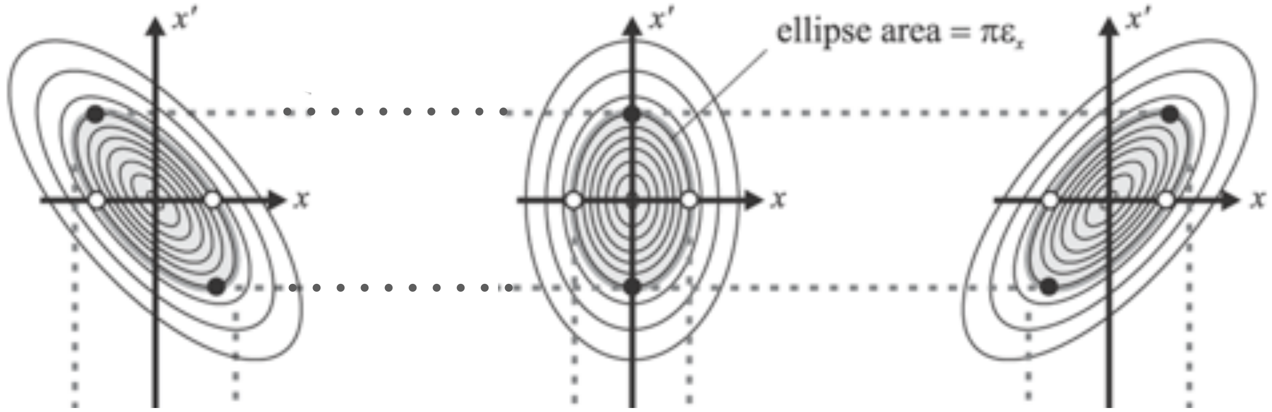


Simple example: F0D0 cell



One complete oscillation in four cells ⟹ phase advance 90° per cell

- ϵ is a constant of the motion, independent of s
- α, β, γ determine the shape of the ellipse in x-x' phase space
- α, β, γ are periodic functions for a ring, so that the ellipse rotates and shears with position s, while its area $\pi\epsilon$ is conserved (no acceleration)



- Phase advance around a ring gives the tune (number of oscillations per revolution) $Q = \dfrac{1}{2\pi}\oint \dfrac{\mathrm{d}s}{\beta(s)}$
- Beam envelope given by maximum value of x:

$$a = A_{max} w(s) = \sqrt{\epsilon}\, w(s)$$

Equation (1) for w then gives the *envelope equation*:

$$a'' + ka - \frac{\epsilon^2}{a^3} = 0$$

The Beta-Function

- Amplitude of a particle trajectory

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos\left(\psi(s) + \phi\right)$$

$$x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos\left(\psi(s)+\phi\right) + \sin\left(\psi(s)+\phi\right)\right\}$$

- Beam envelope

$$a = \sqrt{\epsilon}\sqrt{\beta(s)}$$

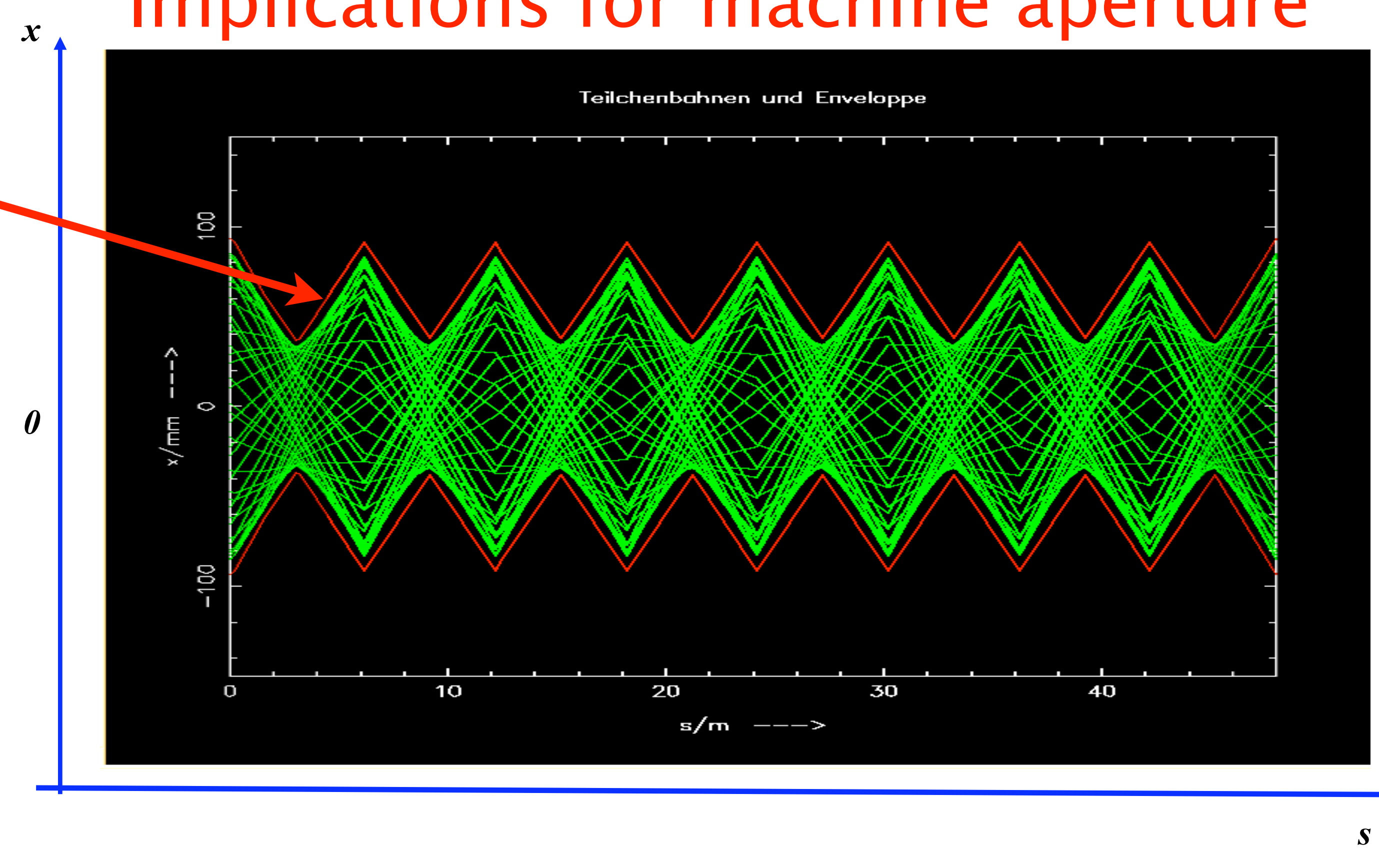
- Courant-Snyder parameters related by:

$$\beta'(s) = -2\alpha(s)$$

$$\alpha'(s) = k(s)\beta(s) - \gamma(s)$$

$$\beta(s)\gamma(s) = 1 + \alpha(s)^2$$

Implications for machine aperture



Transfer Matrix

- From Hill's equation:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos\left(\psi(s) + \phi\right)$$

$$x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos\left(\psi(s) + \phi\right) + \sin\left(\psi(s) + \phi\right)\right\}$$

- For a lattice periodic over a distance L, can express in matrix form:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s+L} = \underbrace{\begin{bmatrix} \cos\Delta\psi_L + \alpha\sin\Delta\psi_L & \beta\sin\Delta\psi_L \\ -\gamma\sin\Delta\psi_L & \cos\Delta\psi_L - \alpha\sin\Delta\psi_L \end{bmatrix}}_{\mathcal{M},\ \text{transfer matrix}} \begin{bmatrix} x \\ x' \end{bmatrix}_{s}$$

- Can construct $\mathcal{M}$ by multiplying matrices for individual elements

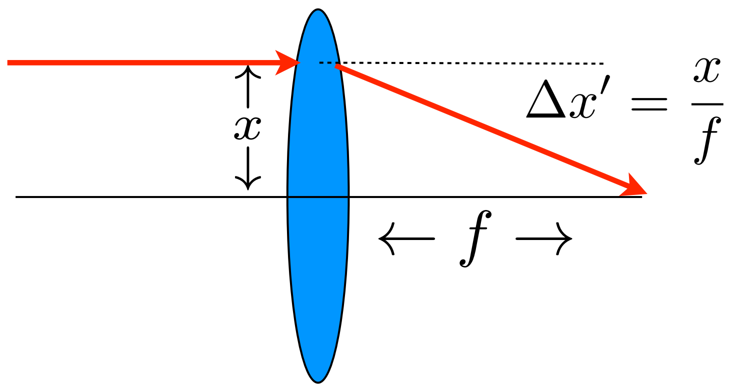
e.g. focusing quadrupole

$$\mathcal{M} = \begin{bmatrix} \cos\sqrt{k}l & \frac{1}{\sqrt{k}}\sin\sqrt{k}l \\ -\sqrt{k}\sin\sqrt{k}l & \cos\sqrt{k}l \end{bmatrix}$$

e.g. defocusing quadrupole

$$\mathcal{M} = \begin{bmatrix} \cosh\sqrt{k}l & \frac{1}{\sqrt{k}}\sinh\sqrt{k}l \\ \sqrt{k}\sinh\sqrt{k}l & \cosh\sqrt{k}l \end{bmatrix}$$

Thin Lens Approximation of F0D0 Cell



Matrix for focusing lens with focal length f is

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

So for F-drift-D, we have

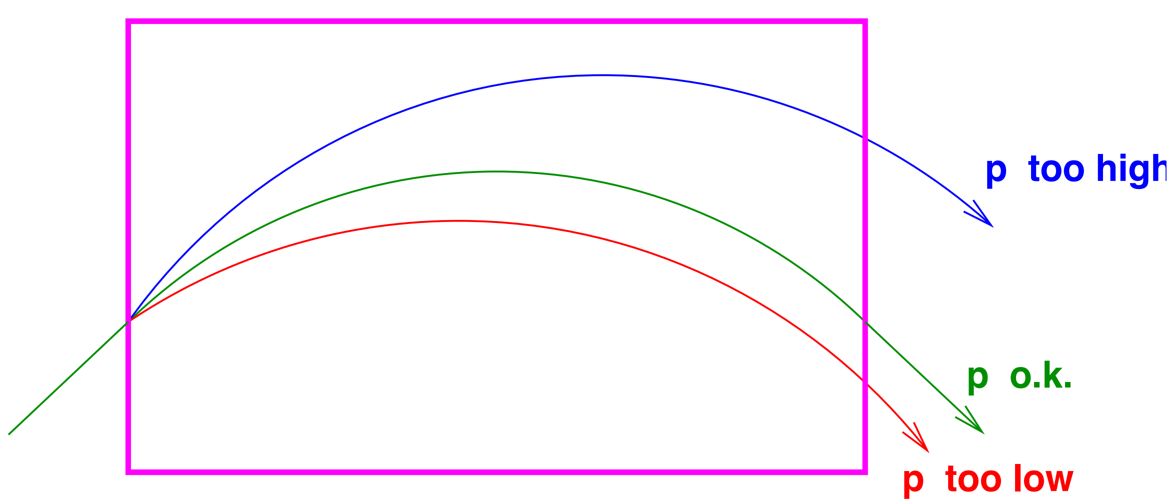
$$\begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{l}{f} & l \\ -\frac{l}{f^2} & 1 + \frac{l}{f} \end{bmatrix}$$

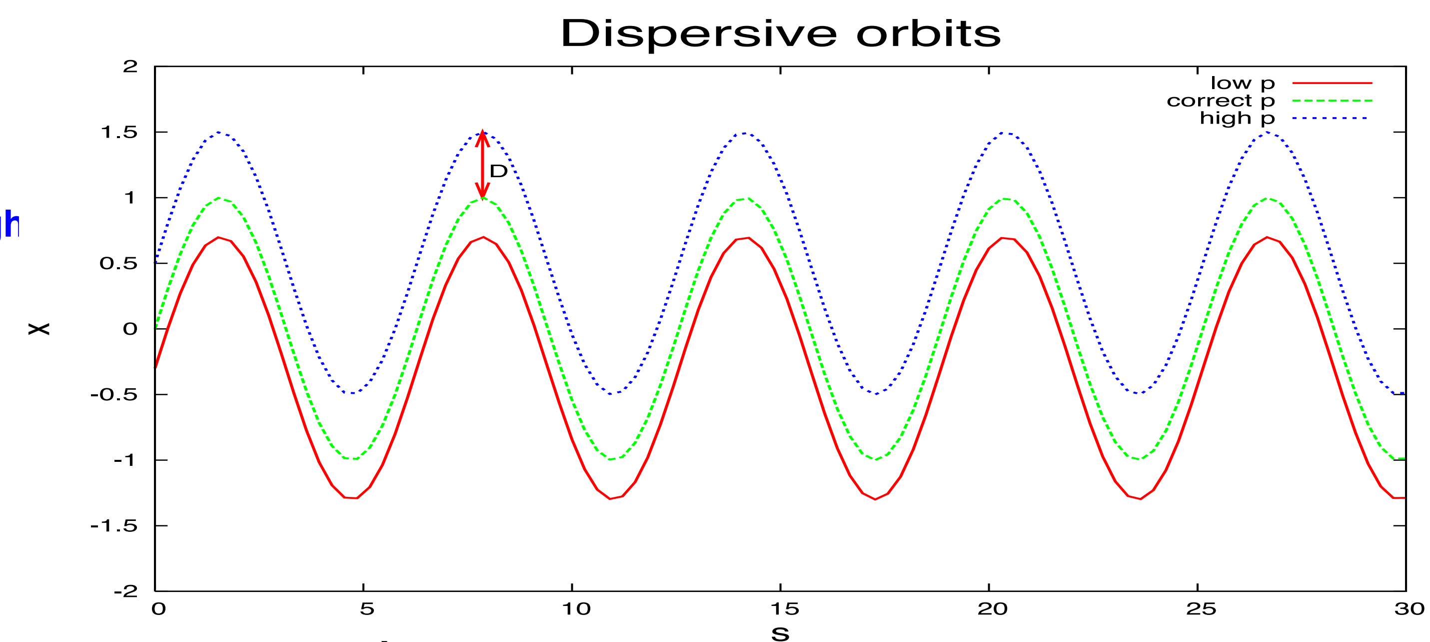
$$\Longrightarrow \quad \begin{bmatrix} x \\ 0 \end{bmatrix}_{\text{in}} \rightarrow \begin{bmatrix} \left(1 - \frac{l}{f}\right) x \\ \left(-\frac{l}{f^2}\right) x \end{bmatrix}_{\text{out}}$$

Equivalent to a thin lens of focal length $\frac{f^2}{l}$, focusing overall if $l < f$. Same for D-drift-F ($f \rightarrow -f$), so system of AG lenses can focus in both planes simultaneously.

Dispersion

- Particles do not all have the same momentum $\Delta p/p \approx 10^{-3}$ (protons), 10^{-2} (muons)
  - particles with different momentum receive different bending and travel on different orbits around the ring





- Equations of motion (see appendix) is

$$x'' + k_x(s)x = \frac{1}{\rho}\frac{\Delta p}{p}$$

$$y'' + k_y(s)y = 0$$

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Dispersion Function

$$x'' + k_x(s)x = \frac{1}{\rho}\frac{\Delta p}{p}$$

Set $x(s) = x_\beta(s) + x_i(s)$ where x_β is the betatron solution.

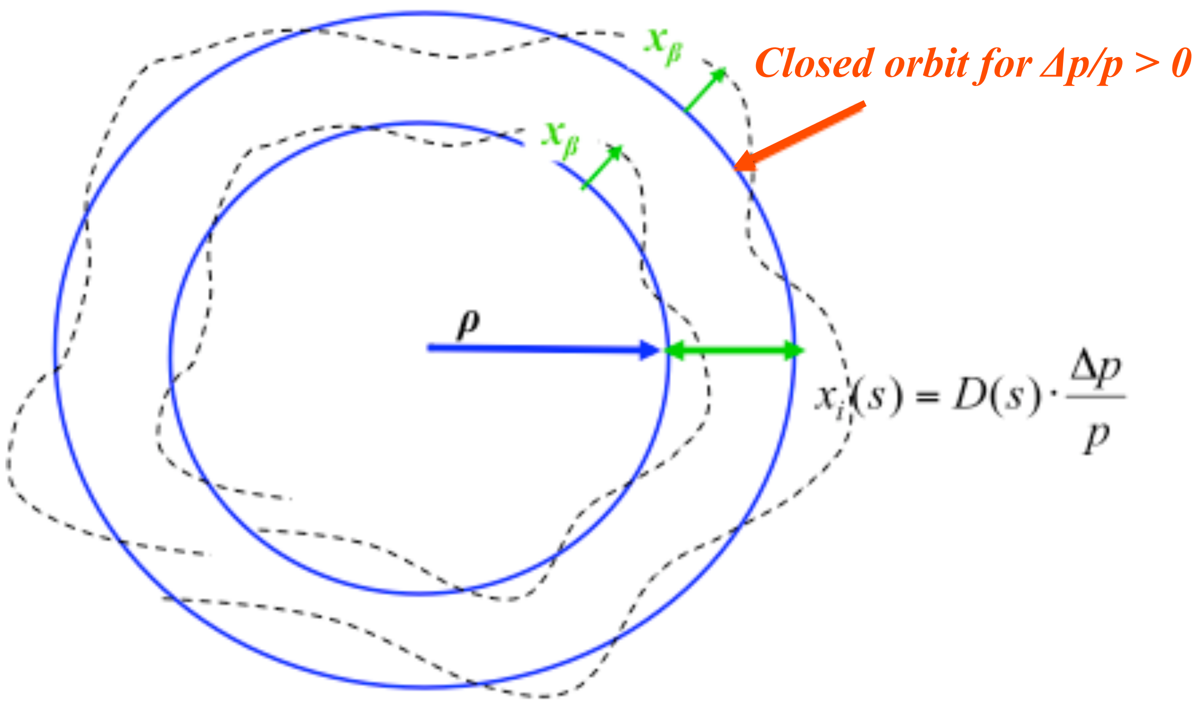
Then $x_i'' + k_x(s)x_i = \frac{1}{\rho}\frac{\Delta p}{p}$ and we define the

dispersion function $D(s)$ by $x_i(s) = D(s)\frac{\Delta p}{p}$

$D(s)$ satisfies $D'' + k_x(s)D = \frac{1}{\rho}$

A factor in setting machine aperture

- Particles perform betatron oscillations about an orbit $D(s)\frac{\Delta p}{p}$
- The dispersion describes the special orbit of a particle with $\frac{\Delta p}{p} = 1$
- The orbit of the particle is the sum of the betatron orbit and the dispersion
- $D(s)$ is subject to the focusing properties of the lattice



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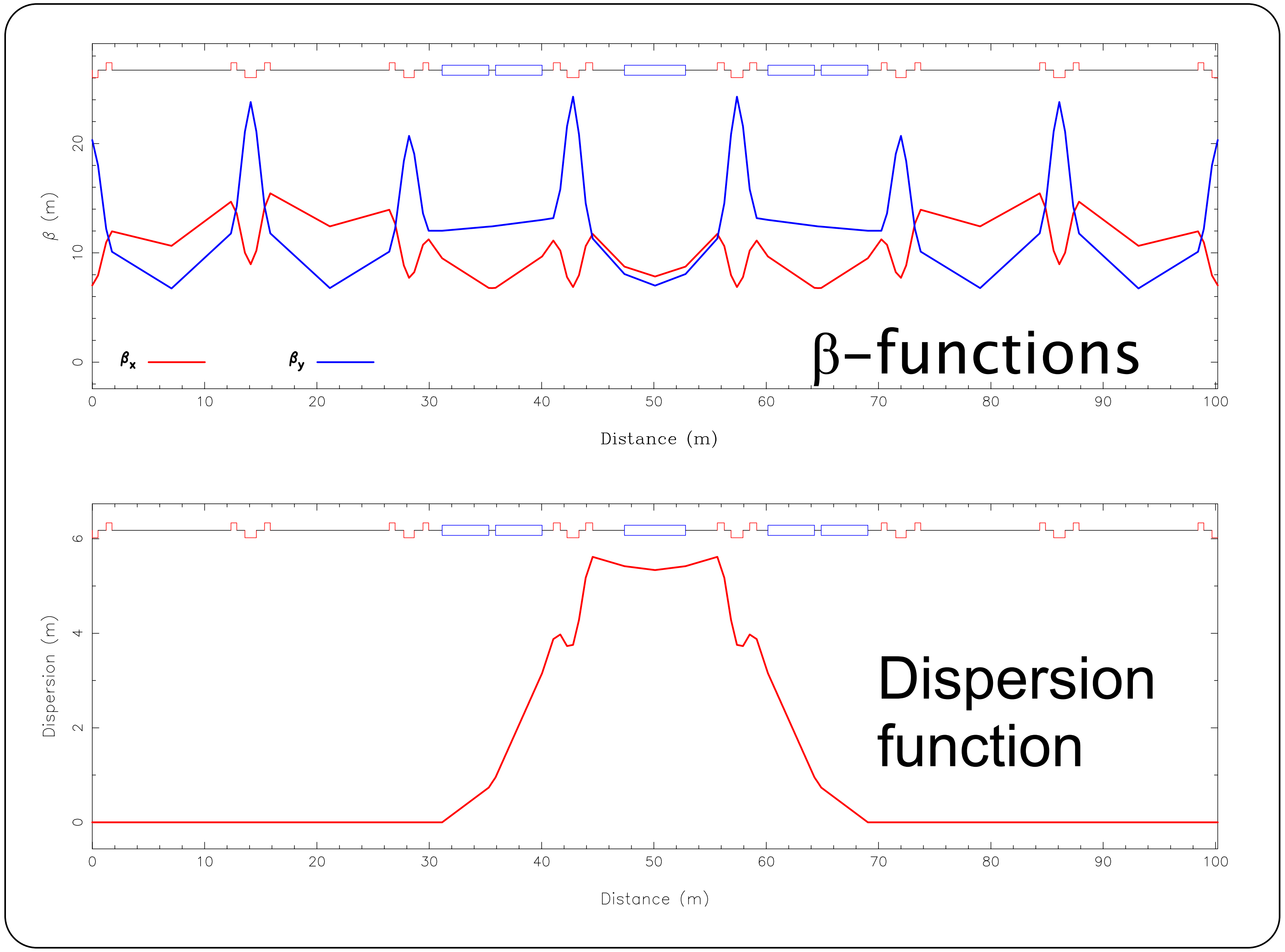
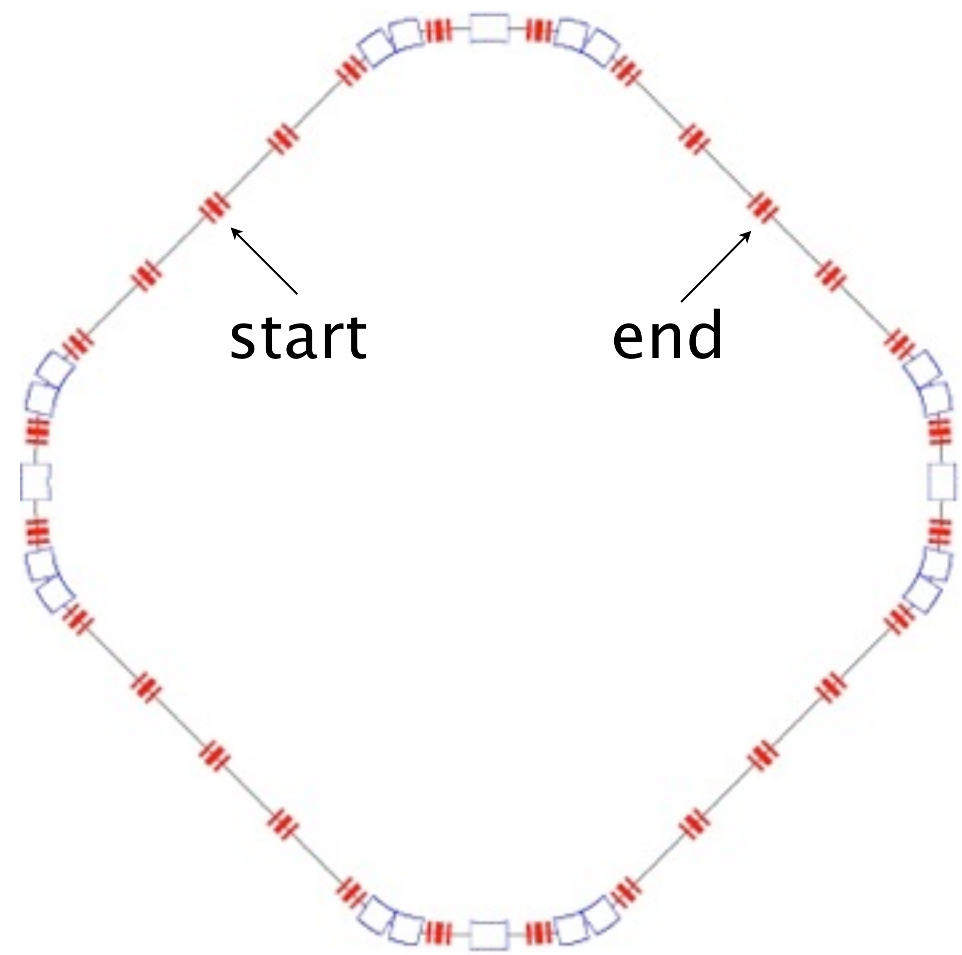
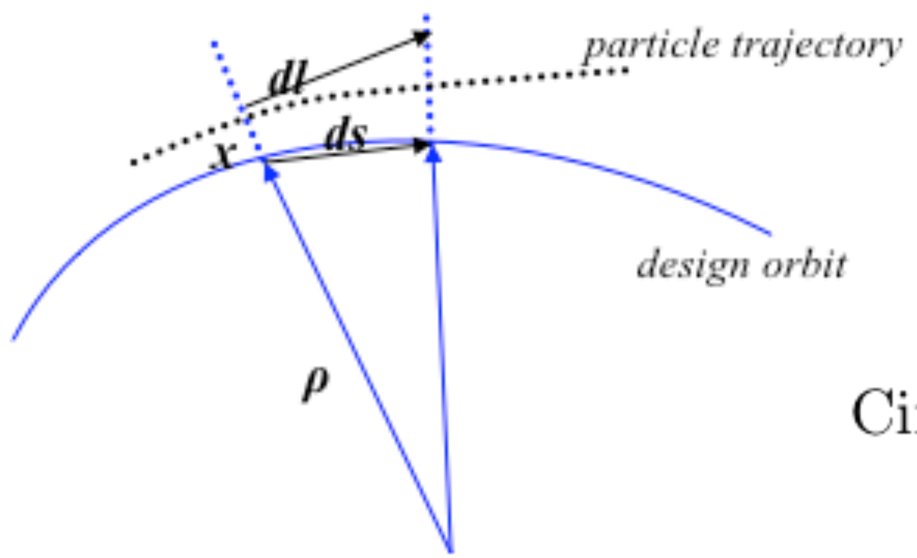
Example: Lattice for Proton Driver for a Neutrino Factory





Momentum Compaction

- Lengthening of the orbit for off-momentum particles



$$\rho \, \mathrm{d}l = (\rho + x) \, \mathrm{d}s$$

$$\Longrightarrow \qquad \mathrm{d}l = \left(1 + \frac{x}{\rho(s)}\right) \mathrm{d}s$$

Circumference of off-energy closed orbit is

$$L_{\Delta E} = \oint \mathrm{d}l = \oint \left(1 + \frac{x}{\rho}\right) \mathrm{d}s$$

Lengthening of off-energy closed orbit is $\delta L_{\Delta E} = \dfrac{\Delta p}{p} \oint \dfrac{D(s)}{\rho} \, \mathrm{d}s$

Definition: $\dfrac{\delta L}{L} = \alpha_p \dfrac{\Delta p}{p}$

$$\Longrightarrow \qquad \alpha_p = \frac{1}{L} \oint \frac{D(s)}{\rho(s)} \, \mathrm{d}s,$$

Momentum Compaction

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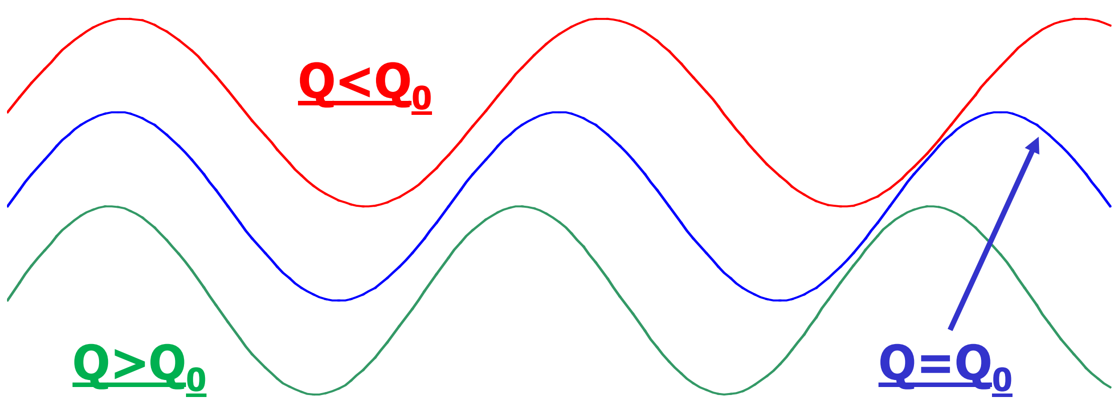
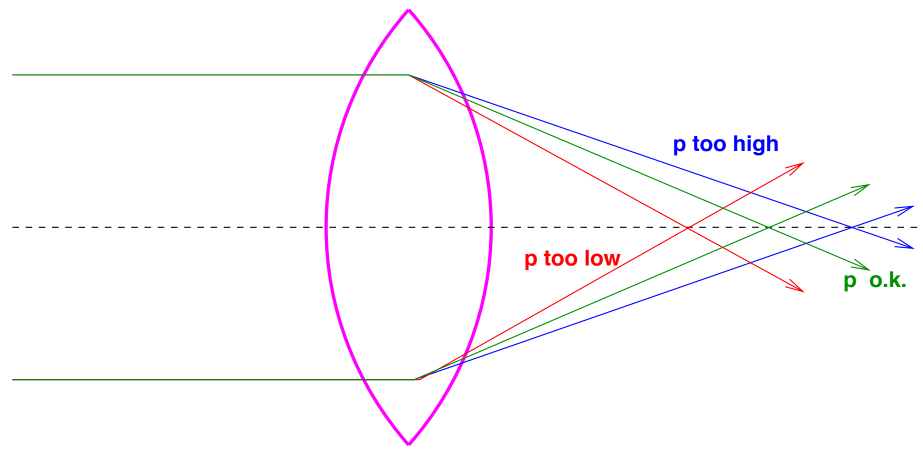
Chromaticity

- Focal length of a quadrupole lens depends on momentum and different focusing leads to different tunes
  - $\Delta p/p>0$ less focusing, reduced Q
  - $\Delta p/p<0$ more focusing, increased Q





$1/f \propto \sin(\frac{1}{2}\mu)$ where μ is phase advance

- The **natural chromaticity** ξ gives the change of tune corresponding to changes in momentum

$$\frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}, \qquad \xi = -\frac{1}{4\pi Q}\oint k(s)\beta(s)\,\mathrm{d}s$$

- Chromaticity is generated by the lattice itself

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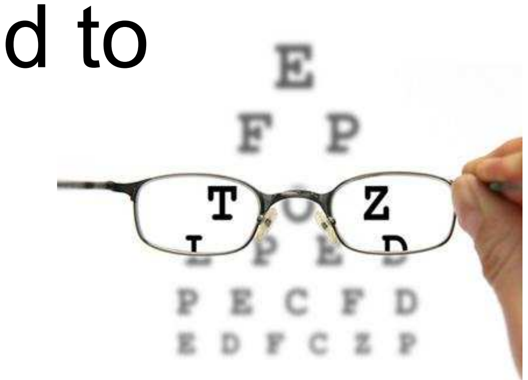
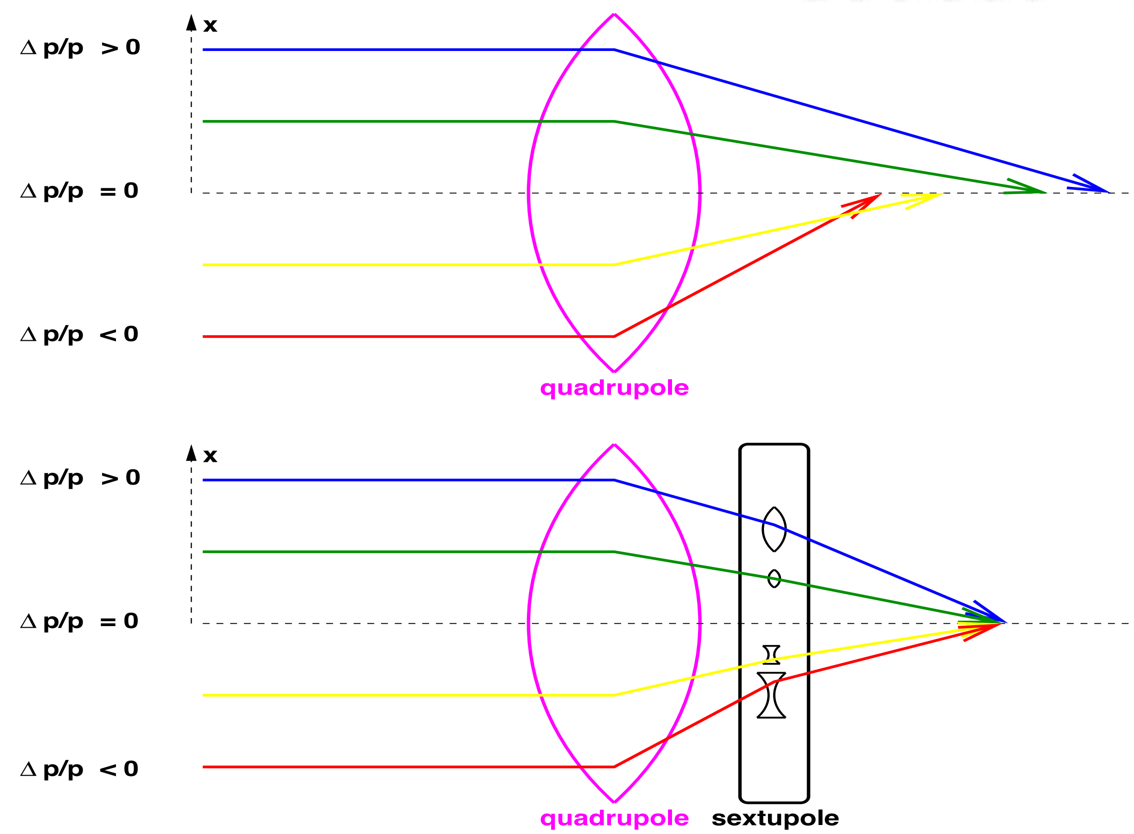
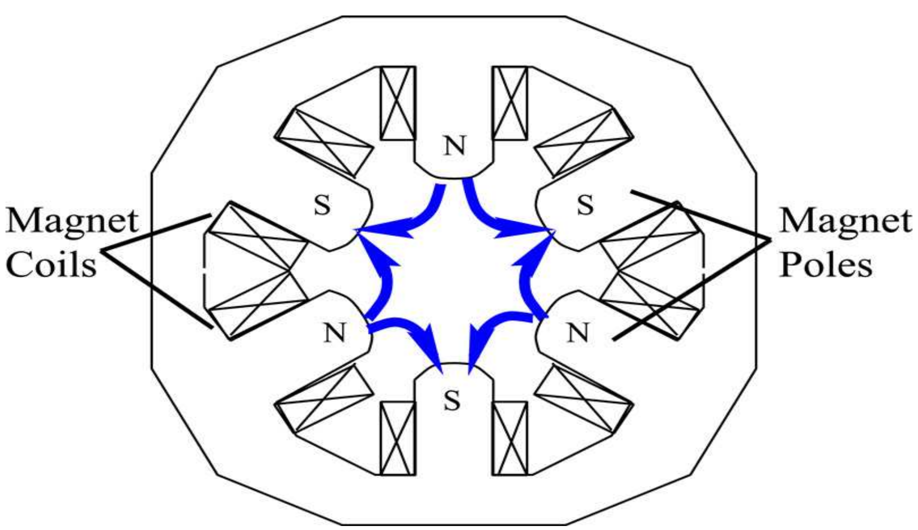
Chromaticity Correction

- Tune spread due to momentum spread can lead to resonances
  - should be as small as possible
- Corrected using sextupoles
  - fields proportion to x^2
  - more focusing for $x > 0$
  - more defocusing for $x < 0$







Transverse Resonances

- May be caused by field imperfections, steering errors etc.
- Change coordinates in Hill's equation (Floquet transformation):

$$\text{Set } \eta = \frac{x}{\sqrt{\beta}} \text{ and } \psi = Q\chi, \text{ so that } \eta(\chi) = \sqrt{\epsilon}\cos\left(Q\chi + \phi\right)$$

- Free betatron oscillation is simple harmonic motion in χ with period 2π

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\chi^2} + Q^2\eta = 0$$

- Need to add in any terms (e.g. from field errors) not taken into account in setting up design orbit:

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\chi^2} + Q^2\eta = -Q^2\beta^{\frac{3}{2}}\frac{\Delta B(\eta, \chi)}{B\rho}$$

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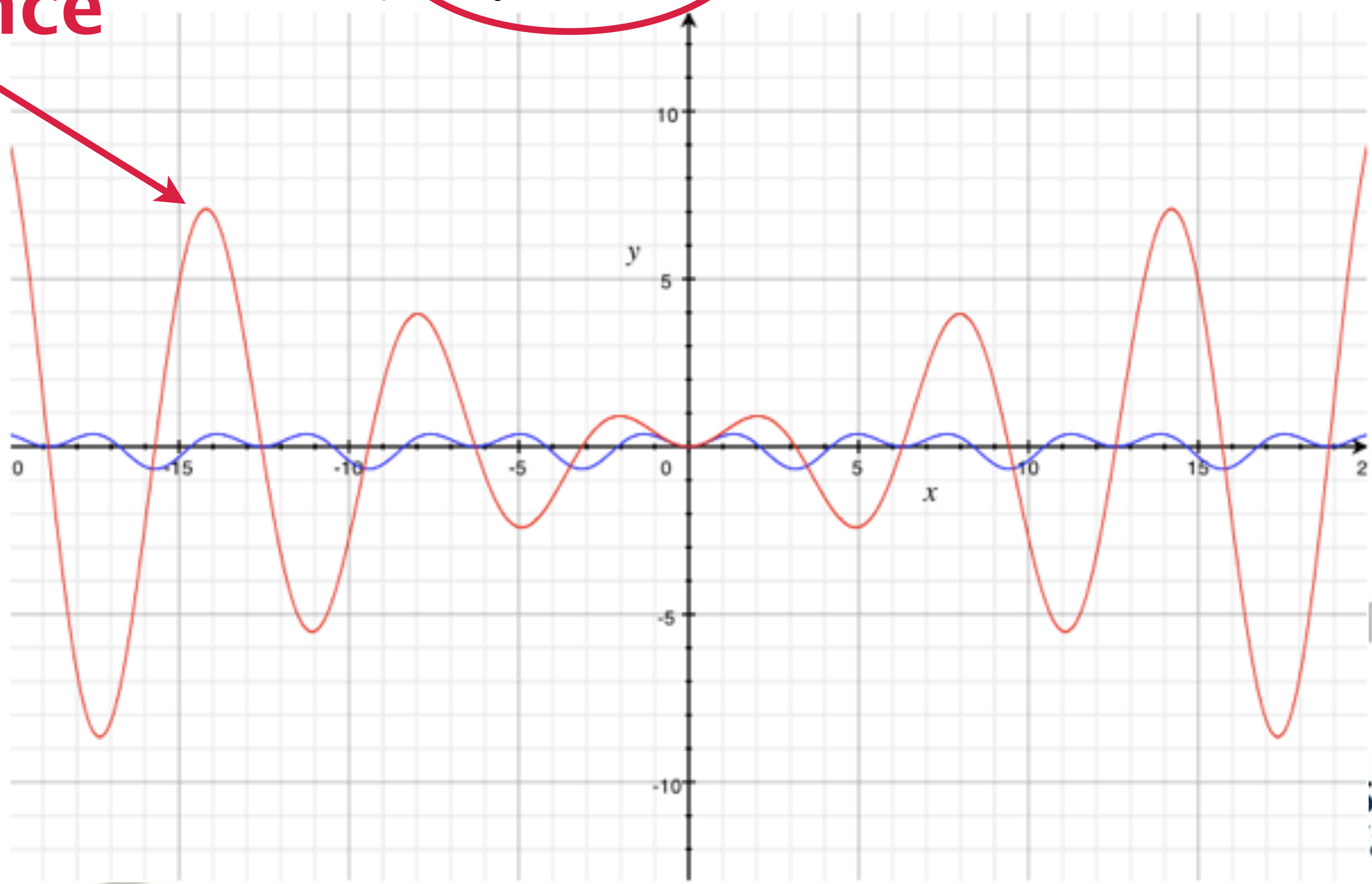
$$\frac{d^2\eta}{d\chi^2} + Q^2\eta = -Q^2\beta^{\frac{3}{2}}\frac{\Delta B(\eta,\chi)}{B\rho}$$

$$\frac{d^2\eta}{d\chi^2} + Q^2\eta = A\cos(\Omega\chi) \text{ has solution}$$

$$\eta = \begin{cases} \frac{A}{Q^2-\Omega^2}\left[\cos(\Omega\chi) - \cos(Q\chi)\right] & Q \neq \Omega \\ \frac{A}{2Q}t\sin(Qt) & Q = \Omega \end{cases}$$

Resonance



$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\chi^2} + Q^2\eta = -Q^2\beta^{\frac{3}{2}}\frac{\Delta B(\eta,\chi)}{B\rho}$$

- Multipole expansion of right-hand side:

$$\Delta B = B_0\left(b_0 + b_1 x + b_2 x^2 + \ldots\right)$$

$$\Longrightarrow \frac{\mathrm{d}^2\eta}{\mathrm{d}\chi^2} + Q^2\eta = -\frac{Q^2 B_0}{B\rho}\left[(\beta^{\frac{3}{2}}b_0) + (\beta^{\frac{4}{2}}b_1)\eta + (\beta^{\frac{5}{2}}b_2)\eta^2 + \ldots\right]$$

Dipole field error. If Fourier series of $\beta^{\frac{3}{2}}b_0$ has non-zero kth harmonic at $k = Q$, a resonant condition will exist. Avoid integer tunes!

Sextupole moments. η^2 can exhibit a frequency $2Q$, which can combine with a non-zero kth harmonic of $\beta^{\frac{5}{2}}b_2$ to produce a resonant condition $k - 2Q = Q$ or $k = 3Q$

Gradient field errors.
kth Fourier harmonic of $\beta^2 b_1$ can beat with frequency Q presented by η to produce driving frequency $k - Q$. Resonant condition is $k - Q = Q$ or $k = 2Q$. Avoid half-integer tunes!

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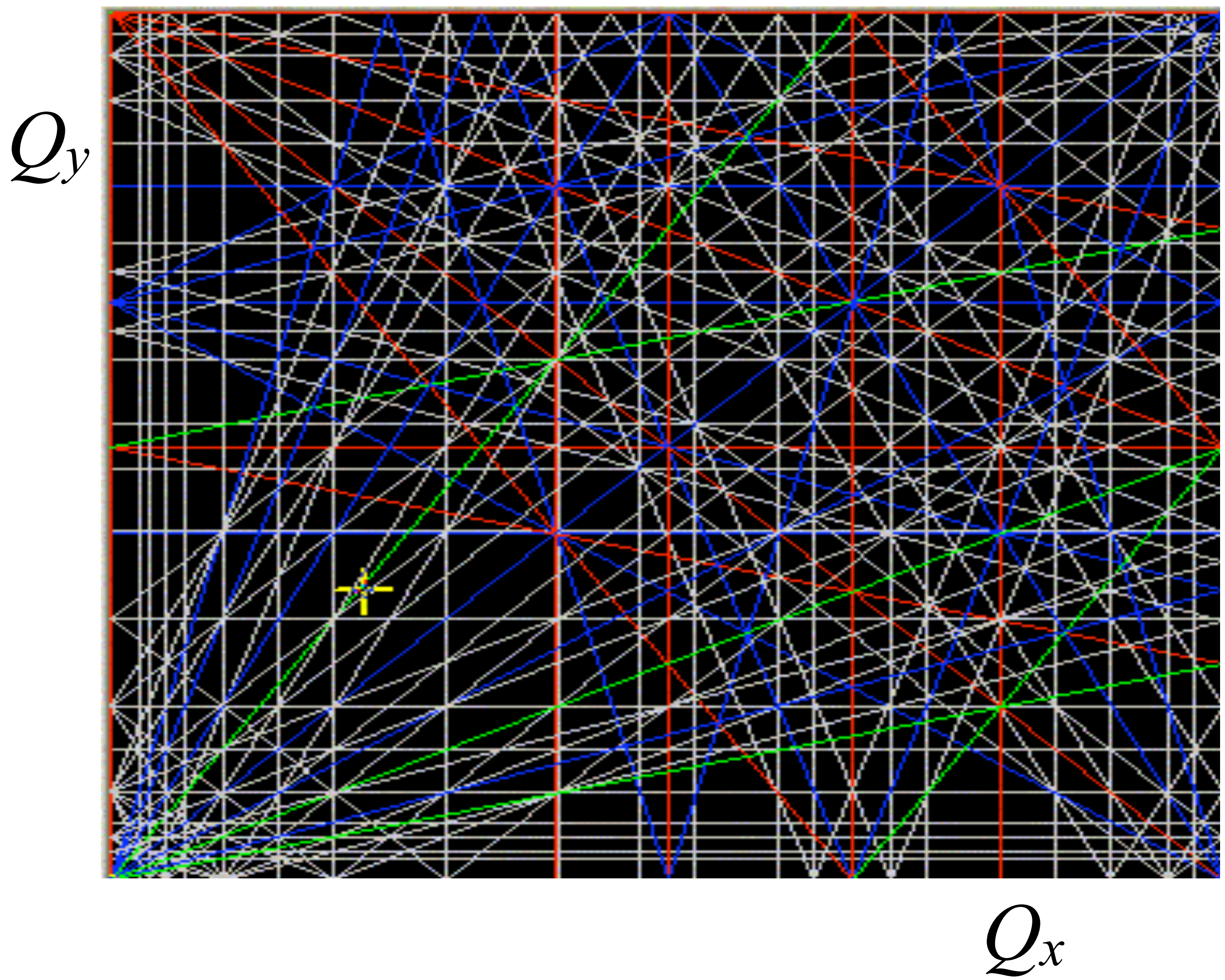
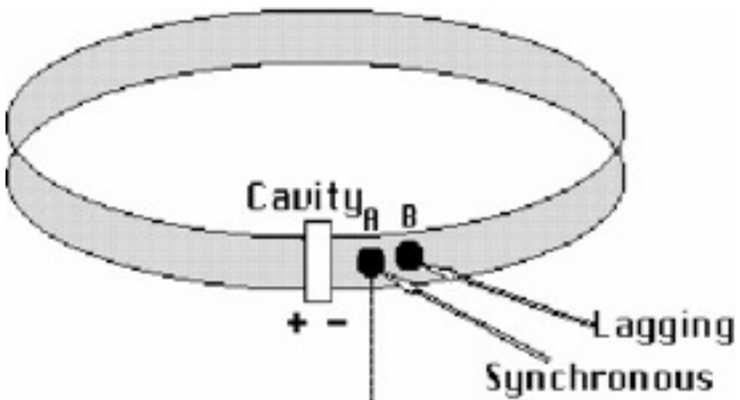
Tune Diagram

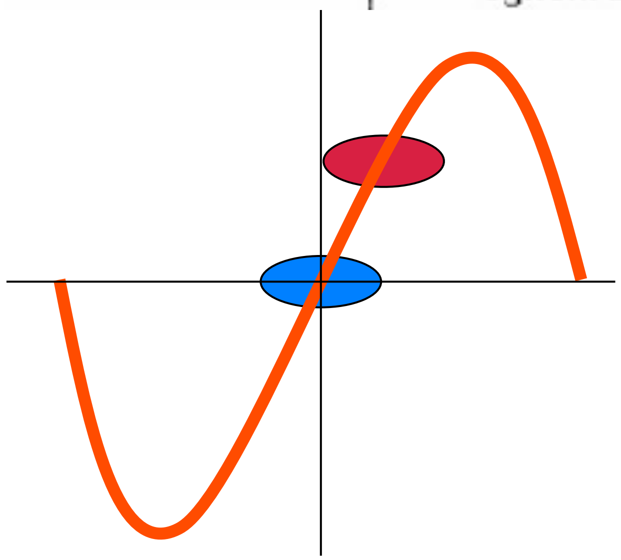
HERA tune plot, up to 3rd order



Resonances will produce beam loss. Challenge: find a place in tune space where the beam will survive

Longitudinal Motion in a Circular Machine: Effect of an RF Cavity





Bunching Effect

- Cavity set up so that particle at the centre of bunch, called the synchronous particle, acquires just the right amount of energy.
- Particles see voltage $V_0 \sin 2\pi\omega_{rf} t = V_0 \sin\varphi(t)$
- Synchronous particle has phase
  - Particles arriving early see $\varphi < \varphi_s$
  - Particles arriving late see $\varphi > \varphi_s$
  - energy of those in advance is decreased relative to the synchronous particle and vice versa.
- To accelerate, make $0 < \varphi_s < \pi$ so that synchronous particle gains energy $\Delta E = qV_0 \sin\varphi_s$

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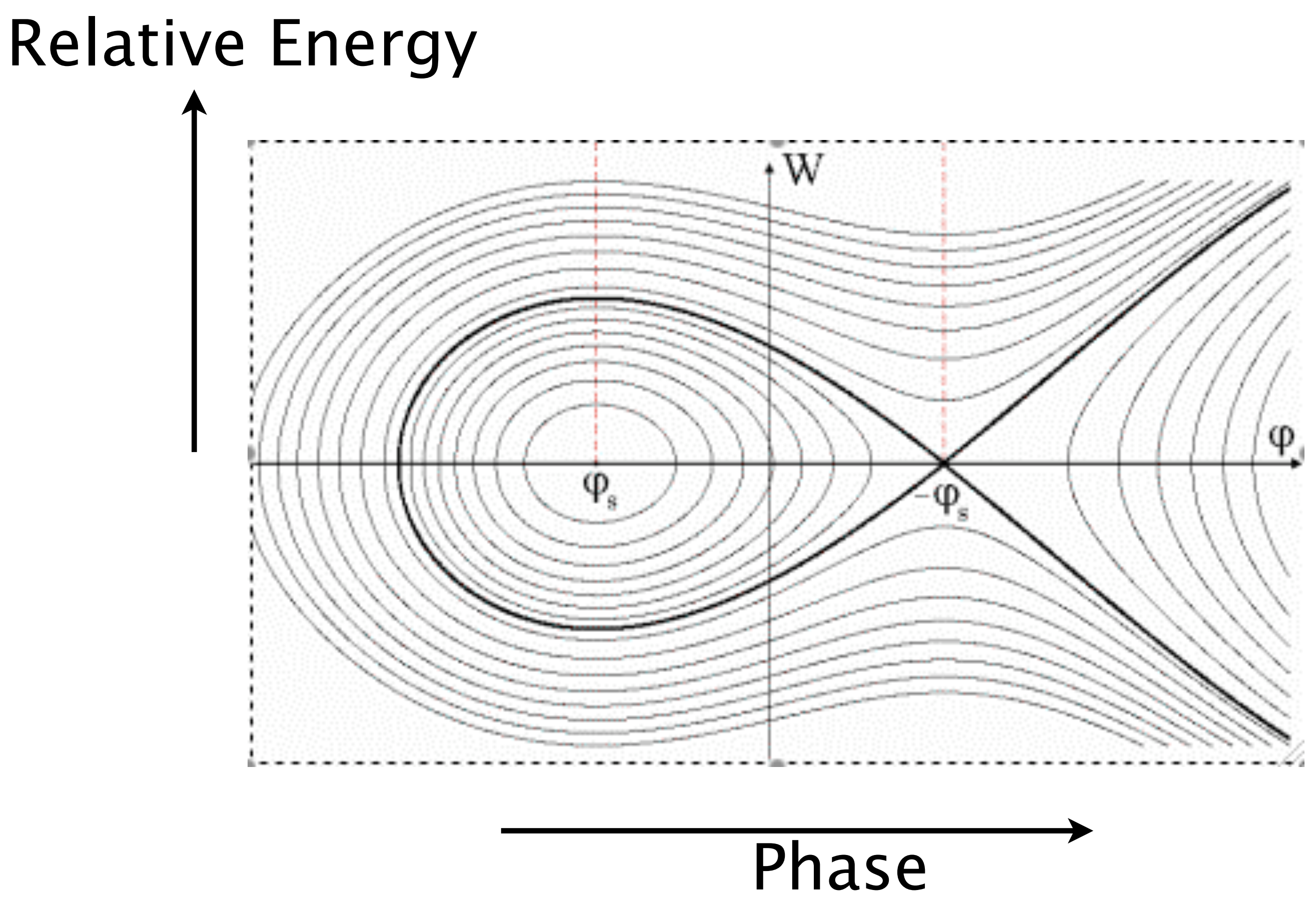
Limit of Stability

- Longitudinal phase space is a useful idea for understanding the behaviour of a particle beam.
- Longitudinally, not all particles are stable. There is a limit to the stable region (the **separatrix** or "bucket") and, at high intensity, it is important to design the machine so that all particles are confined within this region and are "trapped".



Monday, 25 October 2010

R_0 = radius of design orbit, $\omega_0 = \beta_0 c/R_0$ = revolution (angular) frequency of synchronous particle, h = harmonic number

The acceleration rate for the synchronous particle is $\dfrac{\mathrm{d}\mathcal{E}_0}{\mathrm{d}t} = \dfrac{\omega_0}{2\pi} qV \sin\phi_s$, where $\mathcal{E}_0$ is the synchronous energy.

Non-synchronous particles have small deviations of rf parameters:

$$\omega = \omega_0 + \Delta\omega, \qquad \phi = \phi_s + \Delta\phi, \qquad \theta = \theta_s + \Delta\theta$$
$$p = p_0 + \Delta p, \qquad \mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}.$$

θ is the azimuthal orbital angle, where $\Delta\phi = \phi - \phi_s = -h\Delta\theta$

$$\text{so} \quad \Delta\omega = \frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta = -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}\Delta\phi = -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

Energy gain per revolution for non-synchronous particle is $qV\sin\phi$

$$\Longrightarrow \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{\omega}{2\pi} qV \sin\phi$$

Thus
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = \frac{1}{2\pi} qV\left(\sin\phi - \sin\phi_s\right)$$

$$\omega = \frac{\beta c}{R} \quad \Longrightarrow \quad \frac{\Delta\omega}{\omega_0} = \frac{\Delta\beta}{\beta_0} - \frac{\Delta R}{R_0}$$

$$= \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{\Delta p}{p_0} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}\right)\frac{\Delta p}{p_0}$$

α_p is *momentum compaction*, $\alpha_p = \dfrac{1}{\gamma_t^2}$, where γ_t corresponds to the *transition energy* $m_0\gamma_t c^2$

Write $\eta = \dfrac{1}{\gamma_t^2} - \dfrac{1}{\gamma^2}$ (*slip factor*)

Then $$\boxed{\frac{\mathrm{d}\phi}{\mathrm{d}t} = -h\Delta\omega = h\omega_0\eta\frac{\Delta p}{p_0} = \frac{h\omega_0^2\eta}{\beta^2\mathcal{E}_0}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)}$$ since $\dfrac{\Delta p}{p_0} = \dfrac{1}{\beta^2}\dfrac{\Delta\mathcal{E}}{\mathcal{E}_0}$

Hamiltonian Formulation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{h\omega_0^2\eta}{\beta^2\mathcal{E}_0}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = \frac{1}{2\pi}qV\left[\sin\phi - \sin\phi_s\right]$$

$\left(\phi, \frac{\Delta\mathcal{E}}{\omega_0}\right)$ are canonical coordinates in longitudinal phase space and we can construct a Hamiltonian to describe the motion:

$$\mathcal{H} = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2\mathcal{E}_0}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)^2 + \frac{qV}{2\pi}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

Hamilton's equations are

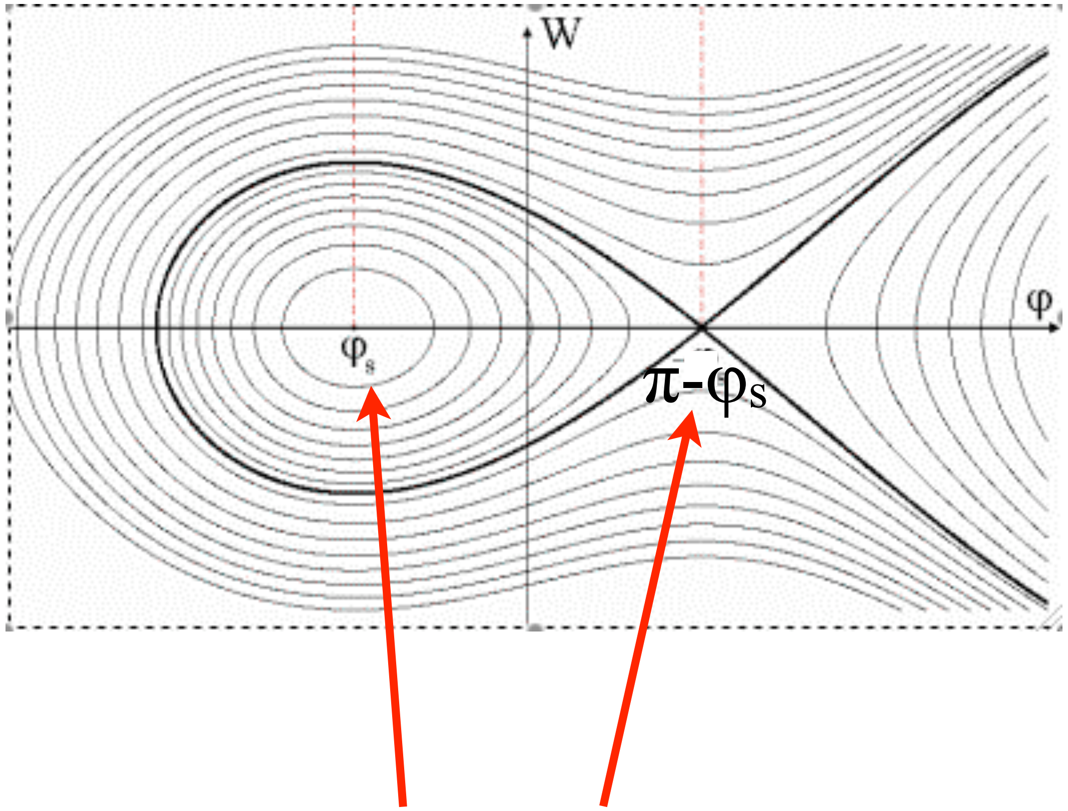
$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial(\Delta\mathcal{E}/\omega_0)} = \frac{h\eta\omega_0^2}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = -\frac{\partial\mathcal{H}}{\partial\phi} = \frac{1}{2\pi}qV\left[\sin\phi - \sin\phi_s\right]$$

Critical Points

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{h\omega_0^2\eta}{\beta^2\mathcal{E}_0}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = \frac{1}{2\pi}qV\left[\sin\phi - \sin\phi_s\right]$$



- Critical points given by

$$\Delta\mathcal{E} = 0, \qquad \sin\phi - \sin\phi_s = 0 \implies \phi = \phi_s, \pi - \phi_s$$

- Separatrix is given by

$$\frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2\mathcal{E}_0}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)^2 + \frac{qV}{2\pi}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

$$= \mathcal{H}_{sep} \equiv \mathcal{H}\left(0, \pi - \phi_s\right) = \frac{qV}{2\pi}\left[(\pi - 2\phi_s)\sin\phi_s - 2\cos\phi_s\right]$$

Stable Fixed Point

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{h\omega_0^2\eta}{\beta^2\mathcal{E}_0}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = \frac{1}{2\pi}qV\left[\sin\phi - \sin\phi_s\right]$$

Combined: $$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\phi - \phi_s) = \frac{h\omega_0^2 qV\eta}{2\pi\beta^2\mathcal{E}_0}\left[\sin\phi - \sin\phi_s\right]$$

linearised $$\approx \frac{h\omega_0^2 qV\eta\cos\phi_s}{2\pi\beta^2\mathcal{E}_0}(\phi - \phi_s)$$

$\Longrightarrow$ stable oscillations for $\eta\cos\phi_s < 0$ (McMillan & Veksler)

Below transition, $\gamma < \gamma_t$, $\eta < 0$, so require $0 < \phi_s < \frac{1}{2}\pi$

Above transition, $\gamma > \gamma_t$, $\eta > 0$, so shift synchronous phase to $\pi - \phi_s$

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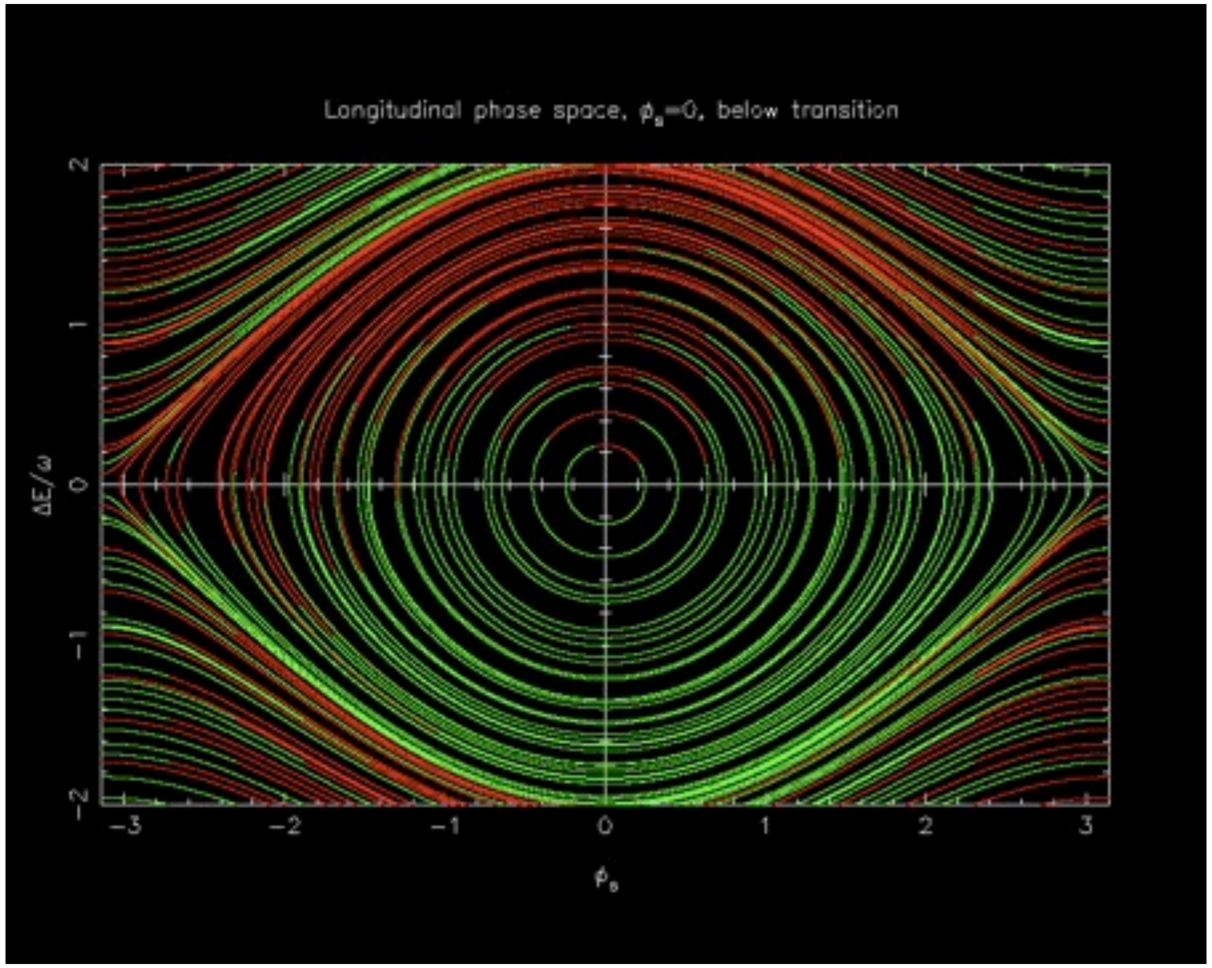
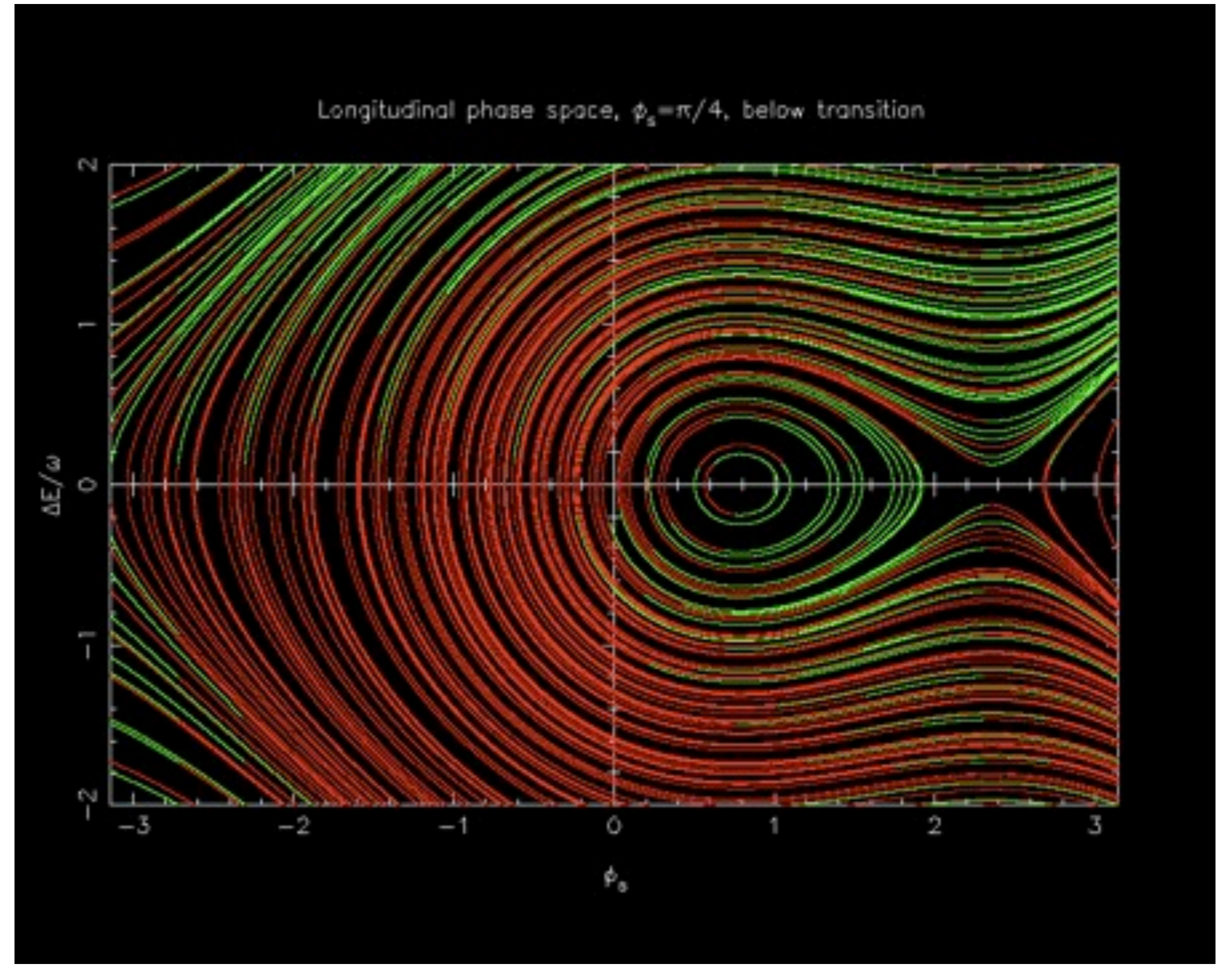
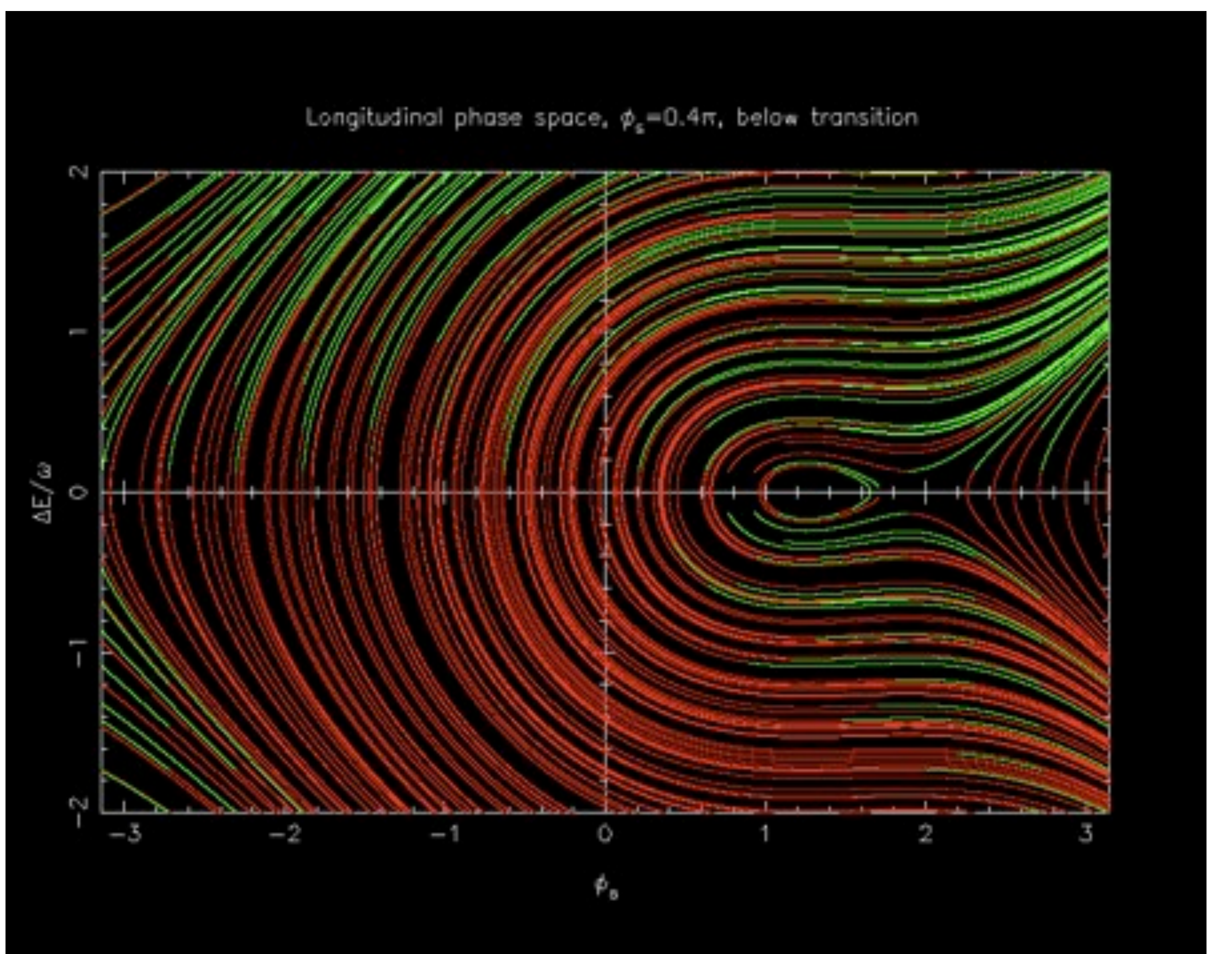
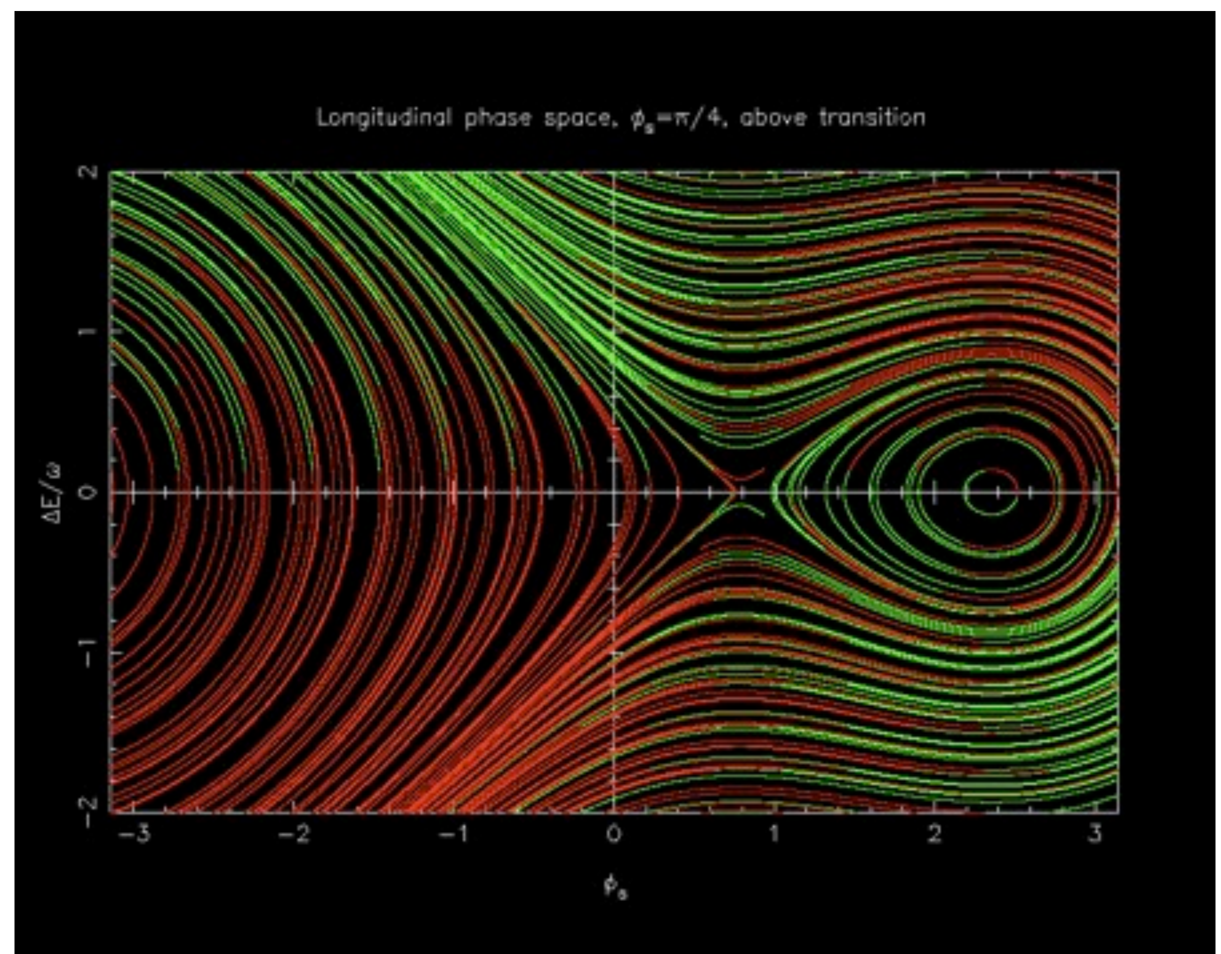

Longitudinal phase space, $\phi_s=0$, below transition
$\Delta E/\omega$
ϕ_s


Longitudinal phase space, $\phi_s=\pi/4$, below transition
$\Delta E/\omega$
ϕ_s


Longitudinal phase space, $\phi_s=0.4\pi$, below transition
$\Delta E/\omega$
ϕ_s

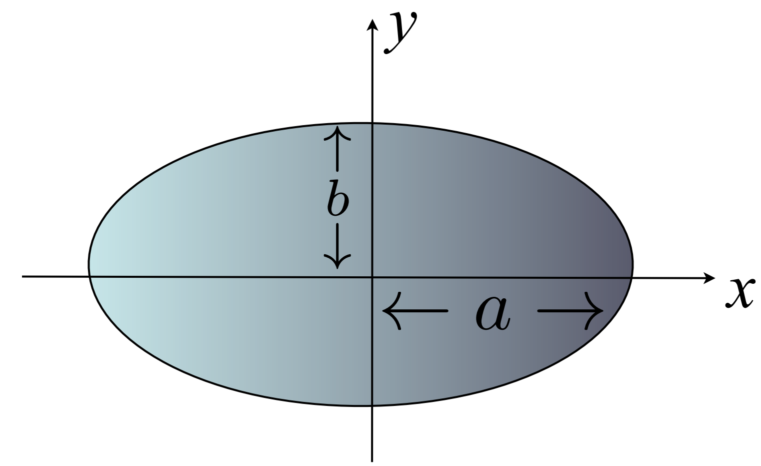

Longitudinal phase space, $\phi_s=\pi/4$, above transition
$\Delta E/\omega$
ϕ_s

Intensity Dependent Effects

- Space-charge: internal (Coulomb) forces between particles.
- Uniform beam produces linear transverse space charge forces:

$$x'' + kx - \frac{\mathcal{K}}{(a+b)}\frac{x}{a} = 0$$

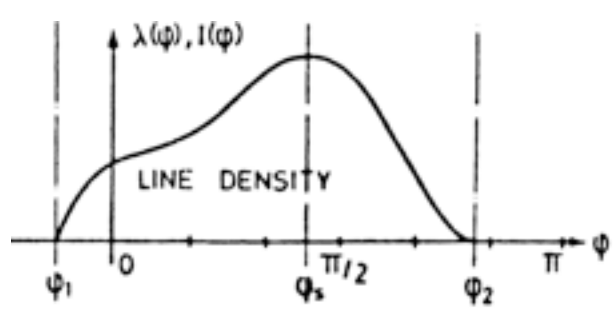
$$a'' + ka - \frac{\epsilon^2}{a^3} - \frac{\mathcal{K}}{a+b} = 0$$

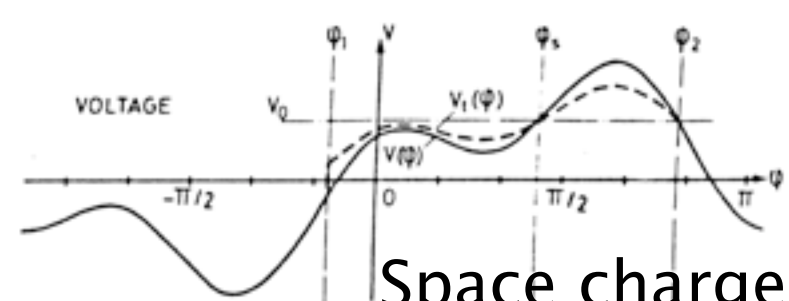


where the *space-charge constant* or perveance is $\mathcal{K} = \frac{I}{I_0}\frac{2}{(\beta\gamma)^3}$

$I_0 = \frac{4\pi\epsilon_0 m_0 c^3}{q}$ is the *characteristic current* ($I_0 \sim 3\times 10^7$ A for protons)

- Space charge produces longitudinal forces $\sim \frac{\mathrm{d}\lambda}{\mathrm{d}s}$ where λ is the line density.





Space charge voltage

Intensity Effects: transverse

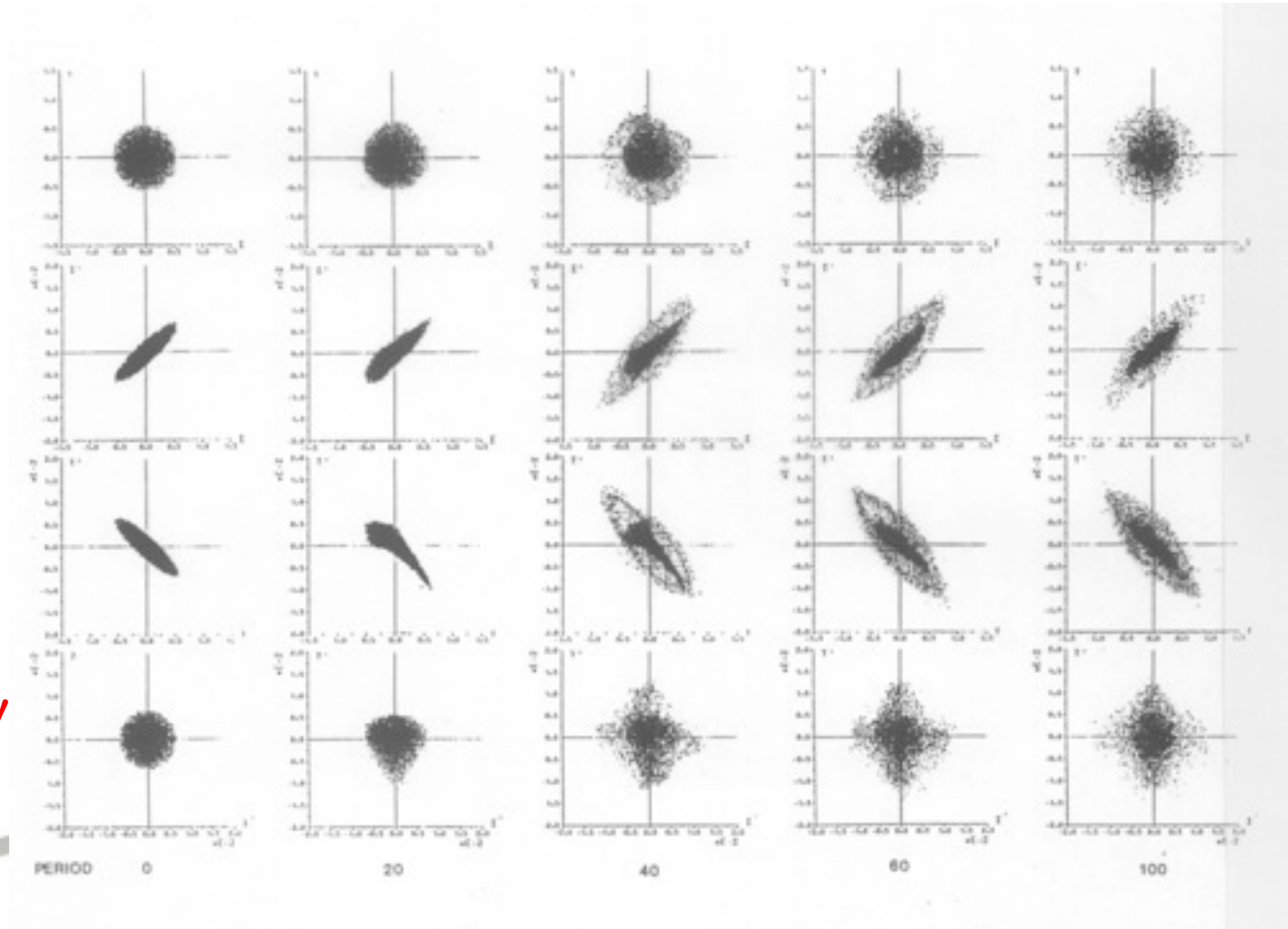
- Transverse space-charge forces lead to
  - reduced focusing and so tune depression and possible resonance

Laslett Tune Shift $$\Delta Q_{sc} = -\frac{N r_0}{2\pi B_f \epsilon \beta^2 \gamma^3}$$

where B_f is the bunching factor $\bar{\lambda}/\hat{\lambda}$

For proton machines aim for $\Delta Q \sim 0.1$

  - non-linear effects such as emittance growth, halo formation and beam loss



Intensity Effects: longitudinal

- Space-charge alters the effective voltage seen by the beam
  - bucket area is reduced
  - could result in beam loss
- A small disturbance in the beam produces a frequency shift
  - Below transition ($\eta<0$), no damaging collective effects
  - Above transition ($\eta>0$), space-charge driven instability, known as the negative mass or microwave instability
  - Can be avoided provided $V_{sp.ch}<0.4\ V_{applied}$; bucket area is reduced by ~23%

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Reading

* E.J.N. Wilson: *Introduction to Accelerators*
* S.Y. Lee: *Accelerator Physics*
* M. Reiser: *Theory and Design of Charged Particle Beams*
* D. Edwards & M. Syphers: *An Introduction to the Physics of High Energy Accelerators*
* M. Conte & W. MacKay: *An Introduction to the Physics of Particle Accelerators*
* R. Dilao & R. Alves-Pires: *Nonlinear Dynamics in Particle Accelerators*
* M. Livingston & J. Blewett: *Particle Accelerators*

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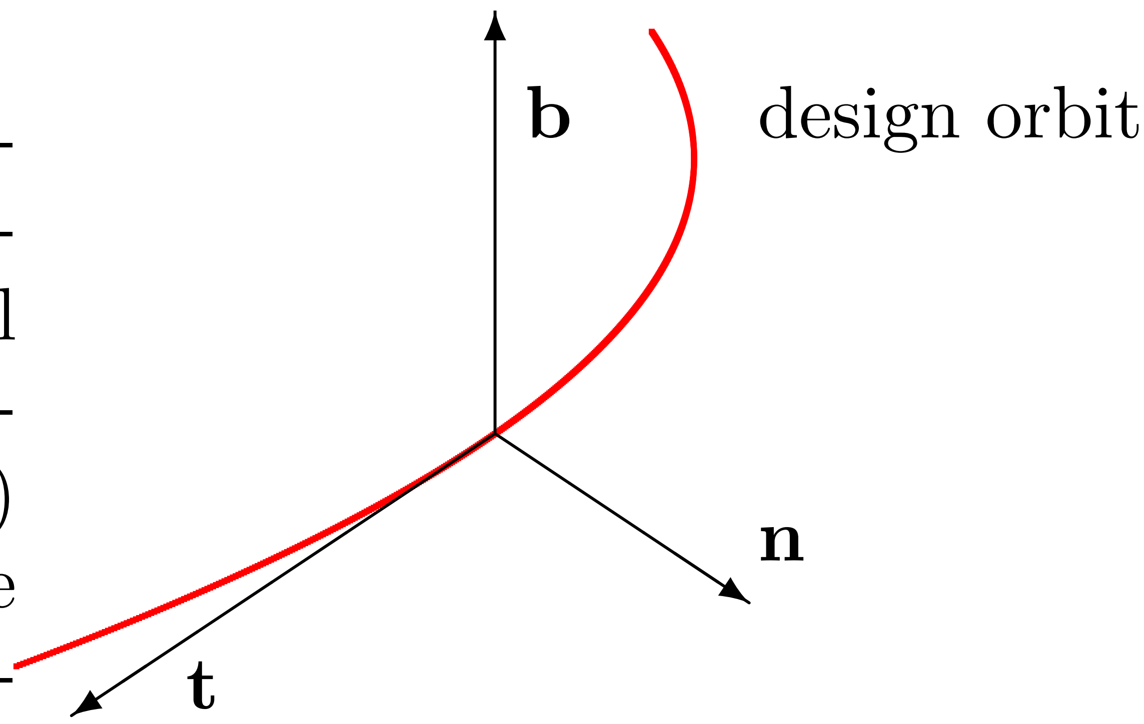
APPENDIX

A particle accelerator is made up of accelerating units, focusing elements, diagnostic equipment, measuring instruments and other special devices. These are assembled together so as to form a channel along which a beam of charged particles can be directed. The channel will comprise straight sections and bends (both horizontal and vertical) and will be calibrated for a specific progression of energies in the beam

An accelerator can be thought of as defining a particular orbit along which a nominal particle having the design energy will travel. This orbit can be given mathematically in terms of arc length s by $\mathbf{r} = \mathbf{r}_0(s)$ and has an associated Serret-Frenet frame of unit vectors in terms of which the motion of a general particle can be described.



The unit tangent to the path is $\mathbf{t} = \dfrac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}$.

Since most accelerators are laid out mainly in a horizontal plane, for a linear channel, we take the unit normal **n** to be horizontal and the unit binormal **b** to be vertical; in the presence of bends, they are defined in terms of the horizontal and vertical curvatures, κ_x and κ_y, by

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa_y \mathbf{b} - \kappa_x \mathbf{n}, \qquad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\kappa_y \mathbf{t}, \qquad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \kappa_x \mathbf{t}.$$

$\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ then form a continuous right-handed triad of moving vectors. Since it would be extremely rare for a beam to undergo horizontal and vertical bends simultaneously, one or other of κ_x or κ_y will be zero, i.e. $\kappa_x \times \kappa_y = 0$.
Particles in general will travel on paths very close to this orbit and have energies close to, but not exactly equal to, the nominal energy. At any

$$\mathbf{r} = \mathbf{r}_0(s) + x\mathbf{n} + y\mathbf{b}.$$

The corresponding velocity is

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{s}\mathbf{t}\left(1 + \kappa_x x - \kappa_y y\right) + \dot{x}\mathbf{n} + \dot{y}\mathbf{b},$$

where the dot denotes $\mathrm{d}/\,\mathrm{d}t$.

The motion of charged particles in electromagnetic fields is governed by the relativistic Lagrangian

$$L = -m_0 c^2/\gamma - e\phi + e\mathbf{v}\cdot\mathbf{A},$$

where $\mathbf{A}$ is tye magnetic vector potential. Since the first fundamental form is

$$\mathrm{d}\mathbf{r}\cdot\,\mathrm{d}\mathbf{r} = \,\mathrm{d}x^2 + \,\mathrm{d}y^2 + h^2\,\mathrm{d}s^2 \qquad \text{where} \quad h = 1 + \kappa_x x - \kappa_y y,$$

the magnetic field $\mathbf{B} = \nabla\wedge\mathbf{A}$ is given by $\mathbf{B} = B_x\mathbf{n} + B_y\mathbf{b} + B_s\mathbf{t}$ where

$$B_x = \frac{1}{h}\left[\frac{\partial}{\partial y}(hA_s) - \frac{\partial A_y}{\partial s}\right] \qquad B_y = \frac{1}{h}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}(hA_s)\right] \qquad B_s = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

Using the formula for $\mathbf{v}$, the Lagrangian becomes

$$L = -m_0 c^2\left\{1 - \frac{1}{c^2}[\dot{x}^2 + \dot{y}^2 + h^2\dot{s}^2]\right\}^{-\frac{1}{2}} + e\left\{\dot{x}A_x + \dot{y}A_y + h\dot{s}A_s\right\} - e\phi.$$

The equations of motion are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0.$$

The corresponding canonical momenta are

$$
\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} = m_0 \gamma \dot{x} + eA_x\,; \\
p_y &= \frac{\partial L}{\partial \dot{y}} = m_0 \gamma \dot{y} + eA_y\,; \\
p_s &= \frac{\partial L}{\partial \dot{s}} = m_0 \gamma \dot{s} h^2 + ehA_s,
\end{aligned}
$$

so that the Hamiltonian governing the motion is

$$
\begin{aligned}
H &= \dot{x}p_x + \dot{y}p_y + \dot{s}p_s - L \\
&= m_0 \gamma c^2 + e\phi \\
&= c\left\{ m_0^2 c^2 + (p_x - eA_x)^2 + (p_y - eA_y)^2 + \left(\frac{p_s}{h} - eA_s\right)^2 \right\}^{\frac{1}{2}} + e\phi.
\end{aligned}
$$

Hamilton's equations then follow from

$$
\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{p_x} = -\frac{\partial H}{\partial x}; \qquad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{p_y} = -\frac{\partial H}{\partial y}; \qquad \dot{s} = \frac{\partial H}{\partial p_s}, \quad \dot{p_s} = -\frac{\partial H}{\partial s}.
$$

Conversion to s as independent variable can be derived by means of Hamilton's principle. In this variational formulation, of which Euler's equations are equivalent to Lagrange's equations,

$$\delta \int_{t_1}^{t_2} L \, \mathrm{d}t = \delta \int_{t_1}^{t_2} \{\dot{x} p_x + \dot{y} p_y + \dot{s} p_s - H\} \, \mathrm{d}t = 0,$$

with all variations vanishing at the end-points t_1, t_2. Rearranging terms, this can be re-written as

$$\delta \int_{s(t_1)}^{s(t_2)} \{x' p_x + y' p_y + t'(-H) - (-p_s)\} \, \mathrm{d}s = 0,$$

where the prime denotes d/ds. The required equations of motion with s as independent variable are then given in terms of Euler's equations with the new Hamiltonian

$$\begin{aligned} K &= -p_s \\ &= -h \left\{ \frac{(H - e\phi)^2}{c^2} - m_0^2 c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2 \right\}^{\frac{1}{2}} - ehA_s \\ &\equiv K(x, y, t, p_x, p_y, -H; s) \end{aligned}$$

Particles deviate only slightly from the design energy, $E_0 = m_0\gamma_0 c^2$, and perform only small oscillations about the design orbit. Using a gauge with $\phi = 0$, define

$$\eta = \frac{H - E_0}{E_0} \qquad \text{so that} \qquad H = E_0(\eta + 1).$$

Between accelerating elements, the design velocity v_0 is constant. Let

$$\sigma = s - v_0 t,$$

which describes the delay of a particle in arrival time at position s. Either by direct substitution or by using the generating functions $F_2(t, p_t) = t(p_t - E_0)$, $F_3(p_t, \sigma; s) = -\frac{1}{v_0} p_t(s - \sigma)$, we derive canonical variables $(x, y, \sigma, p_x, p_y, p_\sigma)$ where

$$p_\sigma = \frac{H - E_0}{v_0}$$

We then have a new Hamiltonian

$$\begin{aligned}
&\mathcal{H}(x, y, \sigma, p_x, p_y, p_\sigma; s) \\
&= \quad p_\sigma + K \\
&= \quad p_\sigma - h\left\{\beta_0^2\left(p_\sigma + \frac{E_0}{v_0}\right)^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2 - m_0^2c^2\right\}^{\frac{1}{2}} \\
&\qquad -ehA_s
\end{aligned}$$

with $\beta_0 = v_0/c$.

Magnetic Elements

Time-independent transverse magnetic fields correspond to a magnetic vector potential $\mathbf{A} = A_s \mathbf{t}$ where

$$\nabla^2 \mathbf{A} = 0 = \nabla \cdot \mathbf{A}.$$

The general solution is a superposition of multipoles but in a particle accelerator elements are designed so that individual terms dominate and other multipoles are present mainly as errors and magnet-end fields. The main types are given by solutions of the form $A_s = \dfrac{1}{n!} A_0 z^n$ where $n \in \mathbb{N}$, $z = x + \mathrm{i}y$ and the real part is understood. Most elements have A_0 real, and elements with A_0 purely imaginary are termed "skew". For straight sections ($\kappa_x = 0 = \kappa_y$), the main magnet types are as follows:

- A *quadrupole* has $n = 2$, A_0 real and provides transverse focusing.

$$A_s = \tfrac{1}{2}G(y^2 - x^2) \qquad \text{where} \qquad G = \left.\frac{\partial B_y}{\partial x}\right|_{x=y=0},$$

  so that the magnetic field is linear with $\mathbf{B} = G(y, x, 0)$.

- A *skew quadrupole* has $n = 2$, A_0 imaginary and is essentially a quadrupole rotated through 45° about the s-axis so that

$$A_s = Nxy \qquad \text{where} \qquad N = \frac{1}{2!}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right)_{x=y=0}.$$

  The field is again linear with $\mathbf{B} = N(x, -y, 0)$.

- A *sextupole* has $n = 3$ and $A_s = \dfrac{1}{3!}\Lambda(x^3 - 3xy^2)$.

  The field is non-linear and $\mathbf{B} = \tfrac{1}{2}\Lambda(-2xy, y^2 - x^2)$.

The most common bending element is a *dipole*, with a constant magnetic field. From the conditions for circular motion, the field required to bend horizontally a particle with nominal momentum $p_0 = m_0 \gamma v$ ($\kappa_y = 0$, $\kappa_x = \kappa$), is $B_x = 0 = B_s$, $B_y = \dfrac{p_0 \kappa}{e}$. Thus, integrating, we have

$$A_s = -\frac{p_0}{2e}\, h = \frac{p_0}{2e}\,(1 + \kappa x).$$

Other fields that might be considered include:

- *combined function magnets*, which combine both dipole bending with quadrupole-type focusing and are given by

$$A_s = -\frac{p_0}{2e}\,h + \tfrac{1}{2}G(y^2 - x^2);$$

- *solenoids* where the magnetic field is symmetric about the s-axis ($B_r = \sqrt{B_x^2 + B_y^2}$, $B_\theta = 0$) and is given by

$$\begin{aligned} B_r(r,s) &= \sum_{i=0}^{\infty} B_{2i+1}(s) r^{2i+1} \\ B_s(r,s) &= \sum_{i=1}^{\infty} B_{2i}(s) r^{2i}. \end{aligned}$$

  In this case the vector potential is purely transverse.

- *general multipoles* corresponding to higher values of n. In particular, $n = 3$ (octupole) and $n = 5$ (dodecapole) often need to be considered, sometimes as individual elements and sometimes as error effects in quadrupoles and dipoles.

Electric Fields

Electric fields provide acceleration and, in particle accelerators, are usually found in the form of radio-frequency cavities.
For a longitudinal electric field, $\mathbf{E} = E(\sigma, s)\mathbf{t}$, and, for cavities oscillating with frequency f,

$$E(\sigma, s) = V(s) \sin\left(2\pi \frac{f\sigma}{v_0} + \phi_0\right).$$

A particle on the design orbit will have $\sigma = 0$ and so acquire an energy $\Delta E = eV(s) \sin\phi_0$. Particles arriving too early will see a reduced field and so will be slowed relative to the design particle; those arriving late will be speeded up. By using a series of cavities with $\phi_0 = 0$, the longitudinal motion of a bunch of particles can be controlled as the natural debunching effect between elements will be counteracted by the longitudinal focusing of the electric fields.

In most magnetic elements, **A** is therefore purely longitudinal and the Hamiltonian can be written as

$$\begin{aligned}\mathcal{H} &= p_\sigma - h\left\{\beta_0^2\left(p_\sigma + \frac{E_0}{v_0}\right)^2 - p_x^2 - p_y^2 - m_0^2c^2\right\}^{\frac{1}{2}} - ehA_s \\ &= p_\sigma - hW - ehA_s,\end{aligned}$$

where

$$\begin{aligned}W &= \left\{\beta_0^2\left(p_\sigma + \frac{E_0}{v_0}\right)^2 - p_x^2 - p_y^2 - m_0^2c^2\right\}^{\frac{1}{2}} \\ &= \sqrt{\beta_0^2\left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2}\left\{1 - \frac{p_x^2 + p_y^2}{\beta_0^2\left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2}\right\}^{\frac{1}{2}}.\end{aligned}$$

The denominator is just

$$\frac{E^2}{c^2} - m_0^2c^2 = p^2,$$

where p is the particle's total momentum.

In an accelerator the total momentum p is very much larger than the transverse momentum components $|p_x|$ and $|p_y|$. We may therefore expand the Hamiltonian in the form

$$\begin{aligned}\mathcal{H} &= p_\sigma - h\sqrt{\beta_0^2\left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2} \\ &\quad \times \left\{1 - \frac{1}{2}\frac{p_x^2 + p_y^2}{\beta_0^2\left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2} + \mathrm{O}\left(\left(\frac{p_x^2 + p_y^2}{p^2}\right)^2\right)\right\} - ehA_s.\end{aligned}$$

The equations of motion are then

$$
\begin{aligned}
x' = \frac{\partial \mathcal{H}}{\partial p_x} &= \frac{h p_x}{p} \\
y' = \frac{\partial \mathcal{H}}{\partial p_y} &= \frac{h p_y}{p} \\
\sigma' = \frac{\partial \mathcal{H}}{\partial p_\sigma} &= 1 - \frac{h\beta_0^2}{p}\left(p_\sigma + \frac{E_0}{v_0}\right)\left\{1 + \frac{1}{2}\frac{p_x^2 + p_y^2}{p^2}\right\} \\
p_x' = -\frac{\partial \mathcal{H}}{\partial x} &= e\frac{\partial}{\partial x}(hA_s) + \kappa_x p\left\{1 - \frac{1}{2}\frac{p_x^2 + p_y^2}{p^2}\right\} \\
p_y' = -\frac{\partial \mathcal{H}}{\partial y} &= e\frac{\partial}{\partial y}(hA_s) - \kappa_y p\left\{1 - \frac{1}{2}\frac{p_x^2 + p_y^2}{p^2}\right\} \\
p_\sigma' = -\frac{\partial \mathcal{H}}{\partial \sigma} &= eh\frac{\partial A_s}{\partial \sigma}
\end{aligned}
$$

In a combined-function magnet, the equations of motion of a charged-particle, linearised in position and momentum, reduce to

$$
\begin{aligned}
p_x &= \frac{px'}{h} \approx p_0 x' \\
p_y &= \frac{py'}{h} \approx p_0 y' \\
p_x' &= -\frac{p_0}{e}\kappa_x h - eGxh + \frac{1}{2}eG\kappa_x(y^2 - x^2) + p\kappa_x \left\{ 1 - \frac{1}{2}\frac{p_x^2 + p_y^2}{p^2} \right\} \\
&\approx -p_0\kappa_x h - eGx + \kappa_x p \\
p_y' &= eGyh \approx eGy.
\end{aligned}
$$

These now give

$$x'' + \left(\frac{eG}{p_0} + \kappa_x^2\right) x = \kappa_x \frac{\Delta p}{p_0} \quad (1)$$

$$y'' - \frac{eG}{p_0} y = 0. \quad (2)$$

Here we have written $p = p_0 + \Delta p$. The effect of a bend is therefore to focus particles in the plane of the bend, and a quadrupole focuses in one direction but defocuses in the other. The quantity p_0/e, which equals $B\rho$, is known as the *magnetic rigidity*, reflecting the fact that particles with a high rigidity require large gradients G for their focusing.