Expanding Field Maps off the mid-Plane

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Use scalar magnetic potential, rather than vector field components.

$$\text{curl } \mathbf{B} = 0 \quad \Longrightarrow \mathbf{B} = \text{grad}\,\phi.$$

Expand $\phi(x, y, z)$ in monomials of increasing order $\{x^i y^j z^k,\ i + j + k = n\}$.

$B_x(x, y, 0) = 0 = B_y(x, y, 0) \implies \phi$ has the form

$$\phi(x, y, z) = \sum_{n=1}^{\infty} \sum_{0 \leqslant i+j<n} \phi_{ij}^n x^i y^j z^{n-i-j} \tag{1}$$

ϕ_{ij}^n are n^{th} order coefficients to be found from the mid-plane data and Maxwell's equations. Note that $\phi_{ij}^n = 0$ for $i + j = n$.

The magnetic field on the mid-plane is

$$B_z(x, y, 0) = \left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \phi_{i,n-i-1}^n x^i y^{n-i-1}.$$

Given $B_z(x, y, 0)$ at a set of points, this gives ϕ_{ij}^n with $i+j = n-1$ by inversion.

Note that the data points can be randomly placed and do not have to lie on a regular grid.

To satisfy Maxwell's equations ($\nabla \cdot \mathbf{B} = 0$), the potential must be harmonic.

$$\nabla^2 \phi = 0 \qquad \Longrightarrow$$

$$(i+2)(i+1)\phi^n_{i+2,j} + (j+2)(j+1)\phi^n_{i,j+2}$$

$$+ (n-i-j)(n-i-j-1)\phi^n_{ij} = 0.$$

The remaining potential coefficients are generated successively from the mid-plane data, for which $i + j = n - 1$.

For each order n there are n such terms, giving a total requirement on data points of

$$\sum_{n=1}^{N} n = \frac{1}{2}N(N+1) \quad \text{for a complete solution up to order } N.$$

Note that since $\phi^n_{ij} = 0$ when $i + j = n$, the recurrence relation tells us that $\phi^n_{ij} = 0$ also when $i + j = n - 2, n - 4, \ldots$.

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Example for third order:

$$\begin{array}{ccccccccccccc}
\phi = & \phi^3_{02} y^2 z & + & \phi^3_{11} xyz & + & \phi^3_{20} x^2 z & + & \phi^3_{01} y z^2 & + & \phi^3_{10} x z^2 & + & \phi^3_{00} z^3 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \text{data} & & \text{data} & & \text{data} & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & & 0 & & 0 & & \text{derived}
\end{array}$$

Code written to invert matrix equation for field-map data:

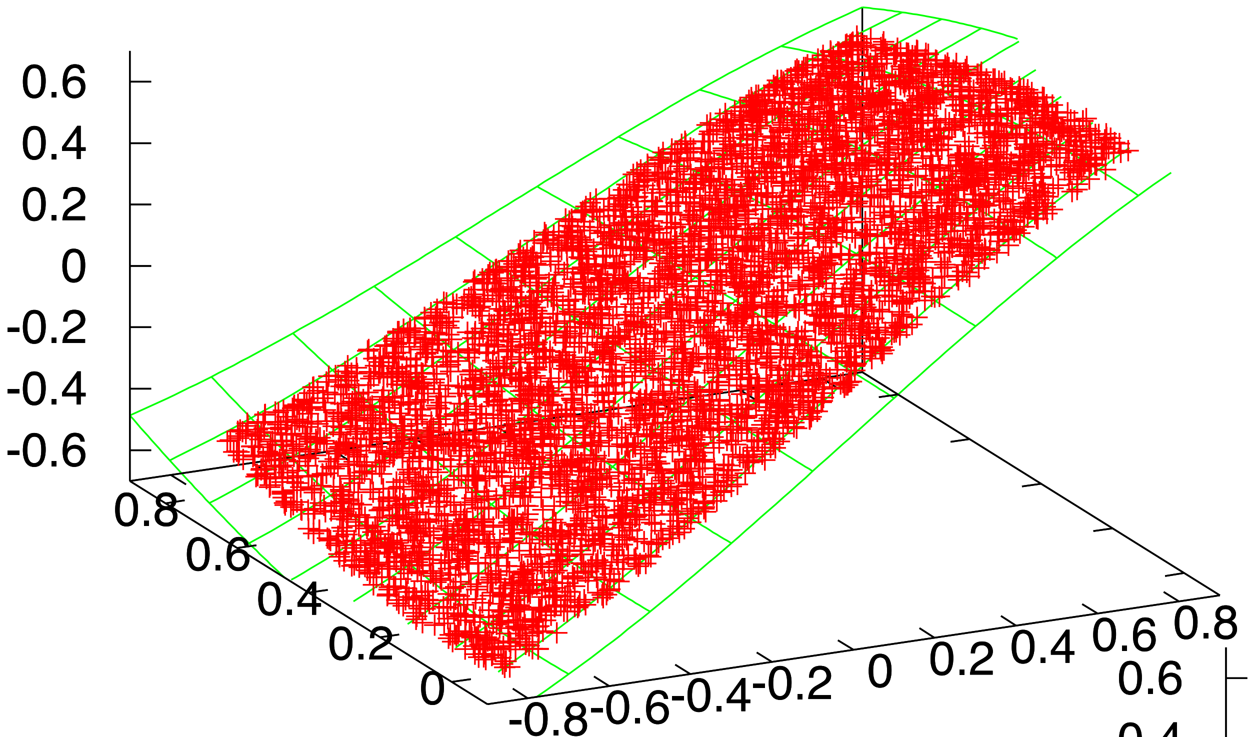
$\mathbf{A}$	$\times$	$\mathbf{\Phi}$	$=$	$\mathbf{B}$
$\downarrow$		$\downarrow$		$\downarrow$
matrix from data coordinates (x, y)		vector of ϕ_{ij}^n with $i + j = n - 1$		vector of field-map B_z values

- Maximum order of monomials given by best fit to number of data points
- Uses LU-decomposition based on standard LAPACK routines with partial pivoting and row interchanges to factor $\mathbf{A}$ as $\mathbf{A} = \mathbf{PLU}$, where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is unit lower triangular, and $\mathbf{U}$ is upper triangular.
- Generates remaining coefficients from recurrence relation.
- A separate module can be called to generate fields at any point (x, y, z) in the map domain.
- A field map in polar coodinates may also be specified.

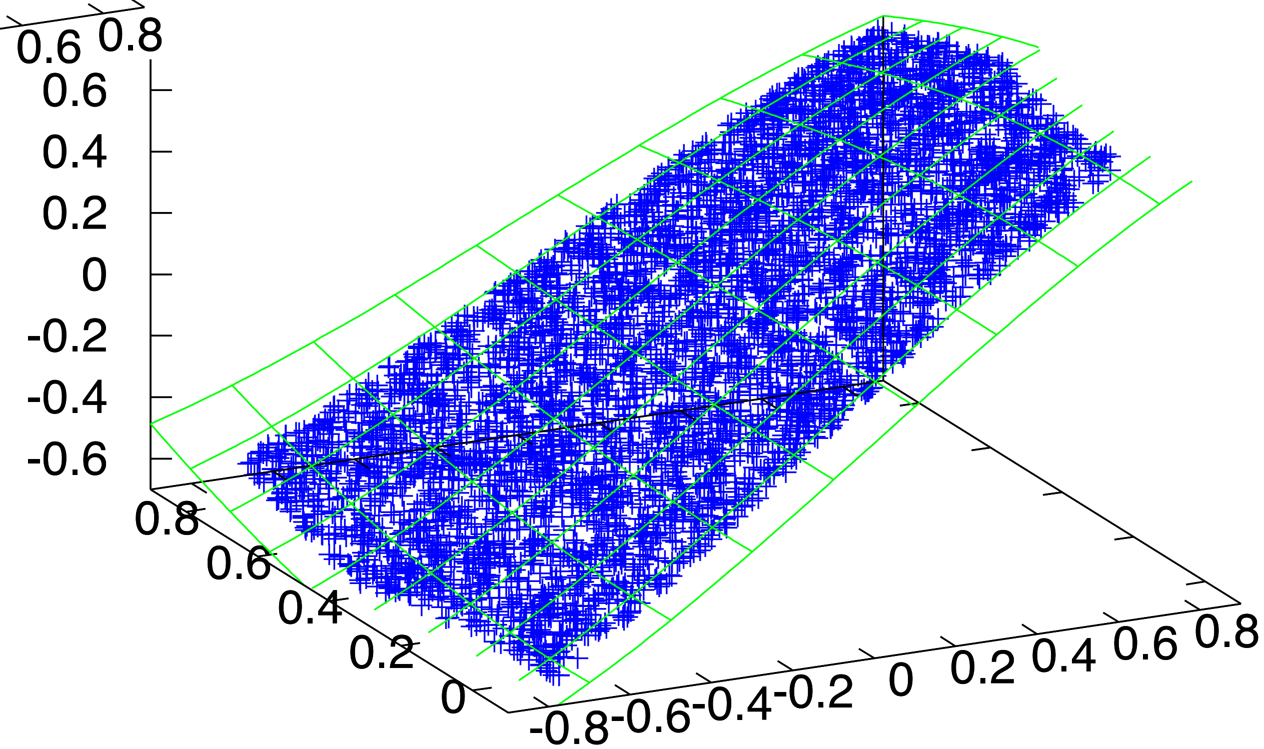
Analytical Model:

$$\phi(x,y,z) = \sin(x)\ \cos(y)\ \sinh(\sqrt{2}z)$$

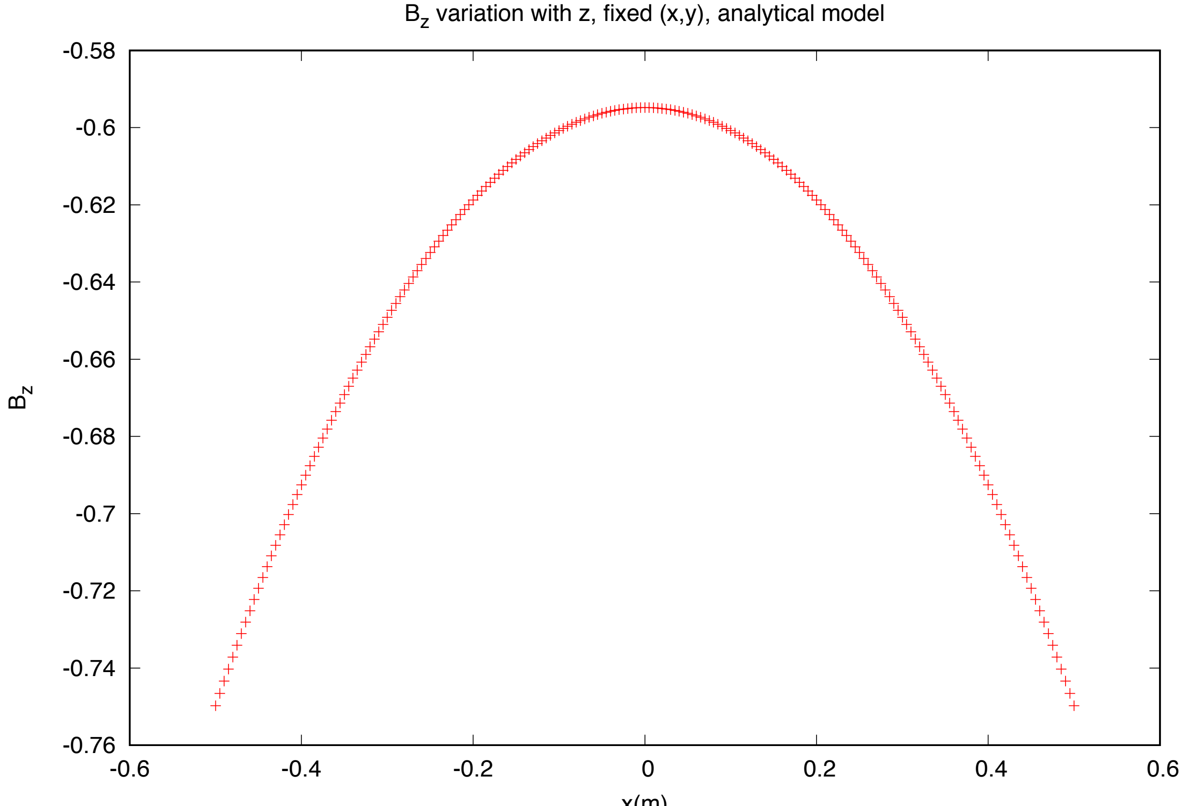
Median-plane B_z



Off Median-plane B_z



B_z variation with z, fixed (x,y), analytical model



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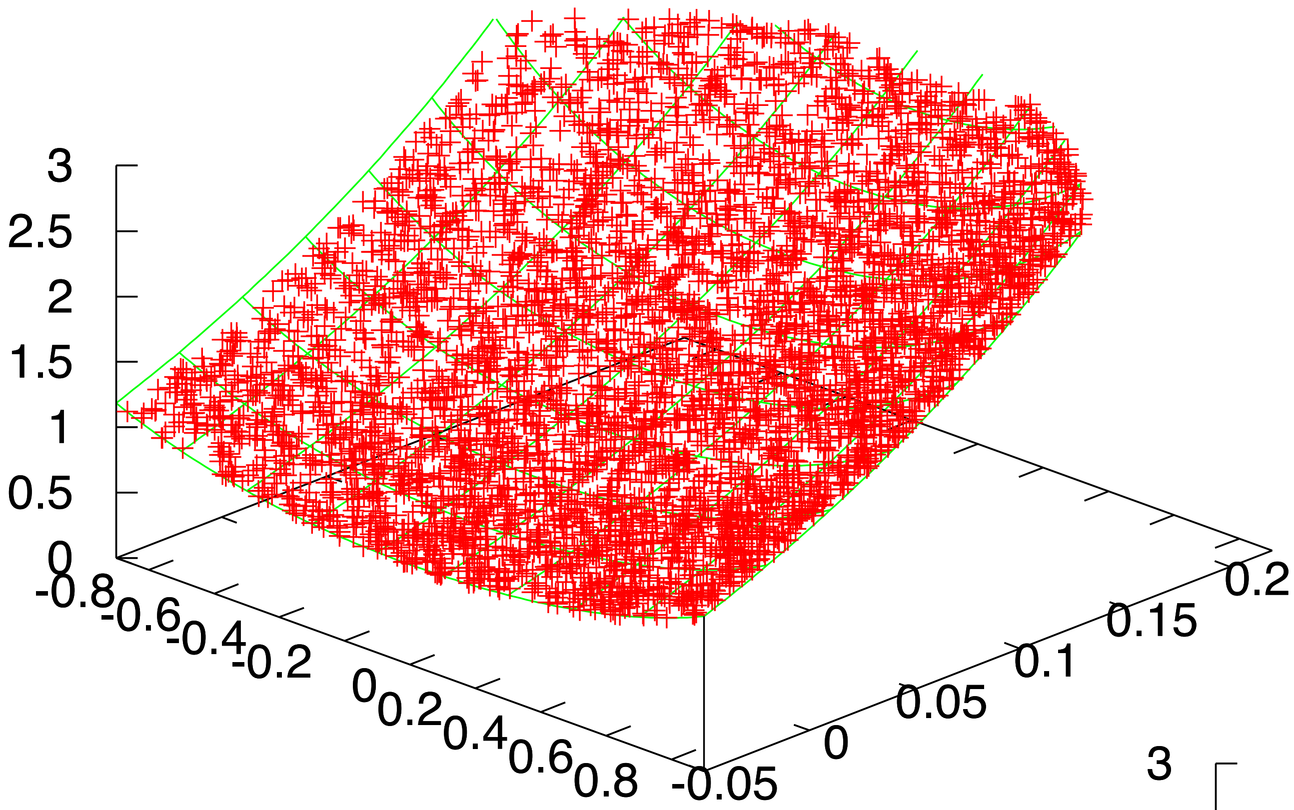
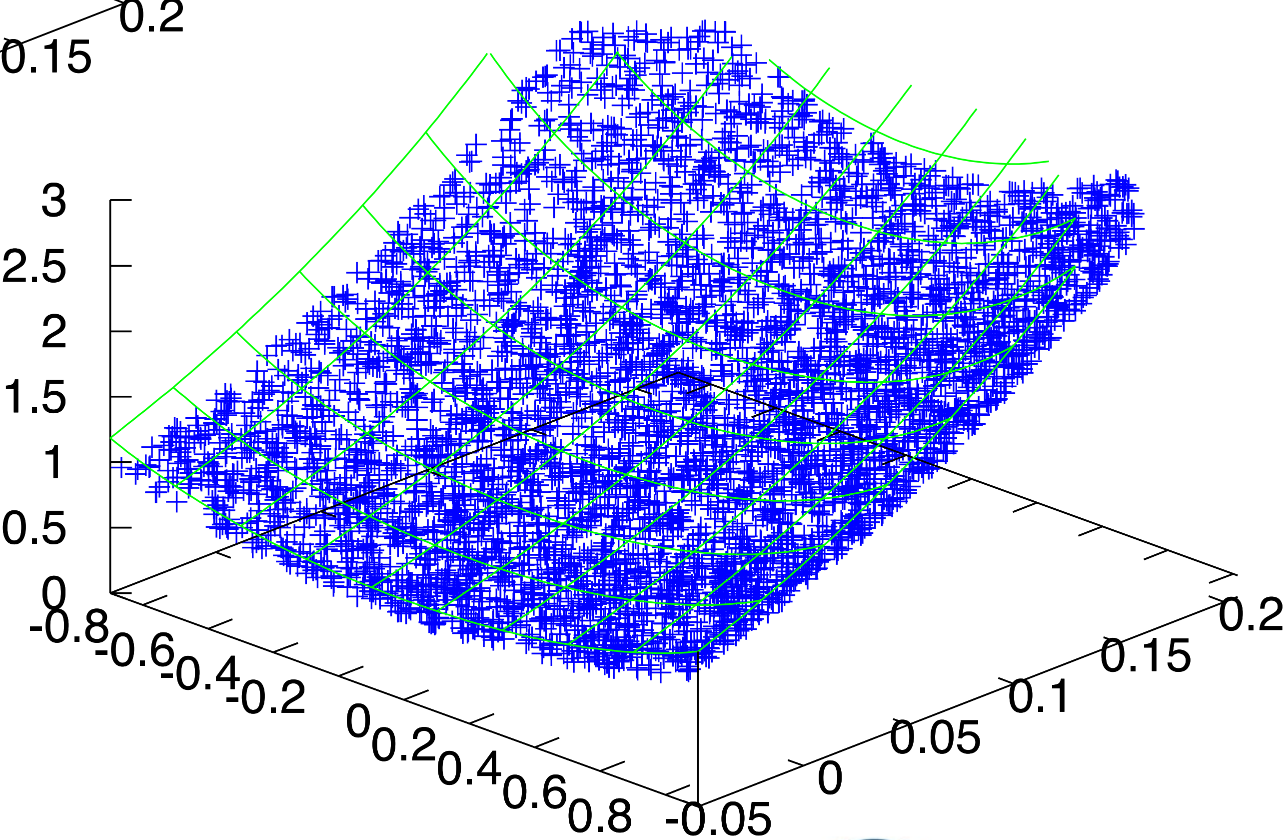
FFAG Model: $B_z = \left(\frac{r}{r_0}\right)^{27}$

FFAG Model: Median-plane B_z



FFAG Model: Off Median-plane B_z



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