Note in response to referee comments

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1 Validity of closed orbit distortion formula

At any fixed momentum, the equation of motion in the horizontal (or vertical) plane in the presence of a dipole kick θ_x is given by

$$x'' + \kappa_1 x = -Re\left[\sum_{n\geq 2} \frac{\kappa_n + ij_n}{n!}\right] + \theta_x \tag{1}$$

where κ_n and j_n are the normal and skew multipole terms, respectively. To obtain the standard formula for the closed orbit distortion (COD) caused by θ_x, the first term on the RHS is neglected, i.e the nonlinearities are ignored. The validity of this approximation is discussed here.

In conventional machines, in which nonlinearities can be ignored, the periodic solution to Eqn. 1 leads to the linear closed orbit response at observable point i caused by a kick at j

$$R_{ij} = \frac{\sqrt{\beta_i \beta_j}}{2\sin\pi q_x}\cos\left(|\psi_i - \psi_j| - \pi q_x\right) \tag{2}$$

The above is reproduced from Eqn. 19 in the paper; the various terms are defined there.

While in general a self-consistent solution to Eqn. 1 cannot be found analytically, an approximate solution can be found by a perturbative approach if the contribution of the nonlinear terms is small. Recalling that the magnetic field in a scaling FFAG varies with Br^k, where k is the field index, the normal multipole components can be written

$$\kappa_n = \frac{1}{B\rho}\frac{d^n B}{dx^n} = \frac{k!}{\rho r^n\left(\kappa - n\right)!} \tag{3}$$

Ignoring the skew components, Eqn. 1 can be written to leading order as

$$x'' + \kappa_x x = -\frac{k(k-1)}{2\rho r^2}x^2 + \theta_x \tag{4}$$

The perturbative approach adopted here is to find the solution in two stages. First the closed orbit is found in the absence of nonlinearities - i.e. find the solution x_0 to

$$x'' + \kappa_x x = \theta_x \tag{5}$$

which satisfies the periodic condition $x_0(s) = x_0(s + C)$ over the circumference C. Next x_0 is substituted into the RHS of Eqn. 6

$$x'' + \kappa_x x = -\frac{k(k-1)}{2\rho r^2} x_0^2 + \theta_x \tag{6}$$

The closed orbit is again found, incorporating the pseudo-kick from the sextupole feeddown $\theta_{sext} = -\frac{k(k-1)}{2\rho r^2} x_0^2$. The question of whether the nonlinearities can be ignored now reduces to the condition

$$\theta_{sext} << |\theta_x| \tag{7}$$

The effect of nonlinearities on the COD can be seen by comparing the result of the Zgoubi tracking code with the prediction of the linear response matrix (Eqn. 2). In figure 1 an example comparison is shown, setting the tune and the dipole kick in the simulation to be reasonably consistent with measurements at the injection momentum. In this case, it can be seen that the contribution of nonlinear terms is well approximated by including sextupole feeddown terms following the perturbative approach described above.

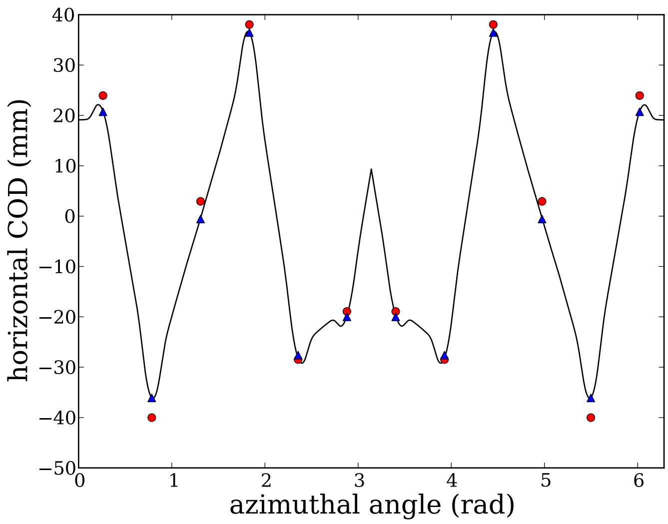

Figure 1: Comparison of the COD found by tracking code (black line), the prediction of the linear closed orbit response (red circles) and the prediction including sextupole feeddown (blue triangles). The symbols are located at the centre of each F magnet. The tune is 3.65 and the dipole kick, located at the cavity (at the midpoint on the horizontal axis), is 50 mrad.

In section 7.2 it is asssessed whether the measurements are consistent with a single dipole error term in the vicinity of the rf cavity. The shape parameter ξ defined as

$$\xi = \frac{r_1(p) - r_7(p)}{r_5(p) - r_1(p)} \tag{8}$$

is introduced as parameter that should depend only on the tune and the location of the error source and is independent of kick angle (so long as just one error source exists in the ring). In fact the assumption that ξ is independent of kick angle is incorrect if nonlinearities are present since the multipole terms act as additional pseudokicks distributed around the ring. Figure 2 shows how ξ, as calculated by a tracking code, varies with kick angle when the tune is set to 3.65 and to 3.85 (corresponding to the injection momentum and to highest momentum at which ξ is measured).

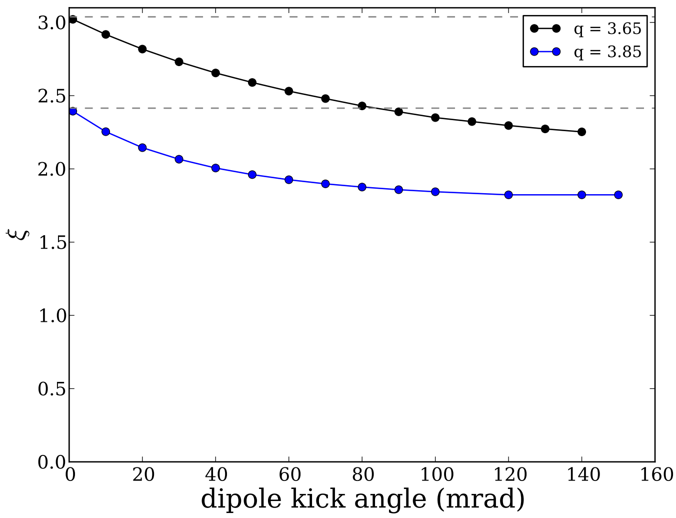

Figure 2: Variation of COD shape parameter ξ a single dipole kick at the location of the cavity as calculated by the Zgoubi tracking code (solid lines). In the absence of nonlinearities, ξ is independent of a single dipole kick (dashed lines). The calculation is carried out with the tune equal to 3.65 (black) and at 3.85 (red).

In section 7.2 it is asserted that the reduction in ξ from 2.8 to 0.93, as calculated from the measured closed orbit data as a function of momentum, is inconsistent with the assumption of a single error source. This assertion should be modified as, once nonlinearities are considered, the observation could be the result of a single dipole kick that grows with momentum.

2 Residual dispersion

The referee wonders if the residual dispersion caused by an RF cavity located at a point of non-zero dispersion could make a significant contributed to the closed orbit. Our calculations show that the effect can be neglected since the synchrotron tune q_s is low ($q_s < 0.005$).

The equation for the residual dispersion is (given by F. Ruggiero (CERN SL/91-38)) can be written as

$$|\Delta D_o| = \sqrt{\beta H_o} \left| \frac{\sin(2\pi q_s)\sin(\pi q)}{cos(2\pi q_s) - cos(2\pi q)} \right| \tag{9}$$

where β is the betatron function at the cavity, q is the betatron tune and H_o is the dispersion invariant given by

$$H_o = \frac{1}{\beta}\left(D^2 + (\beta D' + \alpha D)^2\right) \tag{10}$$

Here ΔD_o is estimated at 145 MeV/c and 360 MeV/c which corresponds to injection and to the highest momentum at which the closed orbit was measured, respectively. As can be seen in Fig. 21 in the draft paper, the measured betatron tunes q at these momenta are approximately 3.6 and 3.85, respectively. In the standard small amplitude approximation, the synchrotron tune is

$$q_s = \sqrt{\frac{heV\eta_0 cos\phi_s}{2\pi\beta_r^2 E}} \tag{11}$$

where β_r is the relativistic quantity. From our RF programme $V = 4kV$, $\phi_s = 30°$, $h = 1$ while the phase slip η_0, as shown in section 3 of the paper, can be written

$$\eta_0 = \frac{1}{\gamma^2} - \frac{1}{k+1} \tag{12}$$

where k is the field index. An approximate value for η_0 is obtained by taking the mean value for k shown in Fig. 9 of the paper. Note, the relatively small variation of k over the momentum range will have little bearing on the synchrotron tune. Finally, at the two momenta mentioned above, q_s is calculated to be 0.0046 and 0.0018, respectively.

In order to simplify the analysis, D' is neglected in the calculation of the dispersion invariant so that $\sqrt{\beta H_o} = D$. Making use of the mean measured dispersion, $D = 0.59$ and the values for q and q_s given above, the residual dispersion $|\Delta D_o|$ is calculated to be 9 mm and 7 mm at 145 MeV/c and 360 MeV/c, respectively.

An upper level on the momentum spread is given by the height of the longitudinal bucket δ_b. Applying the standard formula

$$\delta_b = \frac{2q_s}{h|\eta|}\hat{Y}(\phi_s) \tag{13}$$

where $\hat{Y}(\phi_s)$ is the bucket height factor, one finds $\delta_b = \pm 0.007$ and $\delta_b = \pm 0.003$ at 145 MeV/c and 360 MeV/c, respectively. It follows that the maximum closed orbit shift caused by the residual dispersion is then 61 microns and 22 microns at two momenta. This is much less than the measured $\sim$ 60 mm closed orbit distortion.

3 Beam size measurement

The beam loss occurs over several hundred turns, much longer than the betatron oscillation period of about 3 turns. Since this implies that the smoothed beam loss data is averaged over betatron phase, the information required for an Abel transform is not available.

For clarity, Δa should be replaced by Δx in Eqn. 13 and in the preceding text. As mentioned above, for the purposes of this measurement, the bunch can be considered to be fully decohered in transverse phase space. The phase space ellipse moves, along the bunch centroid, with dispersion along the x-axis (D'=0 at the location of probe). In this case it is clear that $\Delta a = \Delta x$.

4 Derivation of Equation 25

Apologies, equation 25 is incorrect. The correct expression is derived starting from the magnetic field profile in a scaling FFAG with field index k

$$B = B_0 \left(\frac{r_i}{r_0}\right)^k \tag{14}$$

The change in magnetic field ΔB caused by an incremental change in field index, $\Delta k = k_i - \langle k \rangle$, is then given by

$$\Delta B = \Delta k \frac{dB}{dk} = \Delta k B_0 \left(\frac{r_i}{r_0}\right)^k \ln \frac{r_i}{r_0} \tag{15}$$

Note, it is assumed here that all parameters other than k are unchanged in magnet i. The ratio of the resulting dipole kick $\theta_{\Delta k}$ and the bending angle θ_i in magnet i is

$$\frac{\theta_{\Delta k}}{\theta_i} = \frac{\Delta BL}{B\rho} / \frac{BL}{B\rho} = \frac{\Delta B}{B} \tag{16}$$

where L, the effective length of the magnet, satisfies $BL = \int Bds$. Substituting Eqn. 14 and Eqn. 15 one obtains

$$\frac{\theta_{\Delta k}}{\theta_i} = \Delta k \ln \frac{r_i}{r_0} \tag{17}$$

Note that by definition $\theta_{\Delta k} = 0$ when $r_i = r_0$. Equation 17 is compared to simulation in Fig. 3.

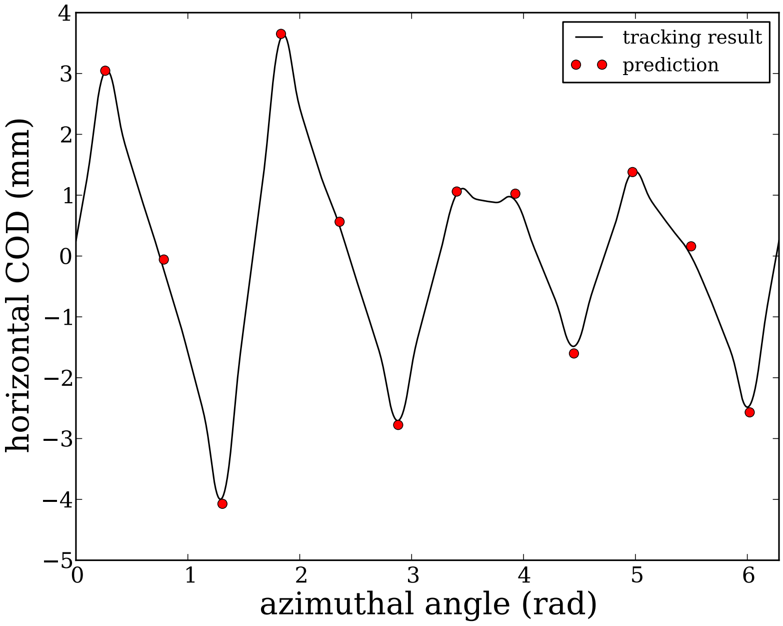

Figure 3: Example COD resulting from a random variation in field index of the F magnets. The variation of the field index is within ±0.1% of the mean. The result of the Zgoubi tracking code (black line) is compared to the COD predicted by the product of the response matrix and the kick calculated using Eqn. 17 (red dots).